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PERKINS' SERIES.

PLANE AND SOLID
G E O M E T R Y.

TO WHICH IS ADDED

PLANE AND SPHERICAL
T R I G O N O M E T R Y
AND
MENSURATION.

ACCOMPANIED WITH ALL THE NECESSARY
LOGARITHMIC AND TRIGONOMETRIC
TABLES.

BY GEORGE R. PERKINS, LL.D.

NEW YORK:
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P R E F A C E.

THIS work is not designed to take the place of the “ELEMENTS OF GEOMETRY,” but rather to give a more extended and comprehensive course, better adapted to the use of Advanced Schools and Colleges.

In the *First Part* of the work, which is confined to the theoretical principles of Geometry, great care has been taken to classify and arrange the various Theorems, by bringing together such as correspond to analogous subjects. We have also arranged the Problems by themselves, and have in no case mixed them up with the Theorems.

In the preparation of the Geometrical portion of this work, we have made free use of the admirable work of VINCENT. The edition used is the *fifth*, revised by its author, with the assistance of the distinguished BOURDON. This is a most excellent and valuable French work, but it is not in all respects well adapted to the wants of our Higher Institutions; we have, therefore, preferred to make selections from it, rather than to give a translation of the whole. We have endeavored to arrange and present the different Propositions in the manner which to us appeared best calculated to impart a thorough knowledge of the most important principles of the science of Geometry. As a general thing, we have adhered to the ordinary method of conducting our demonstrations; still we have not hesitated to introduce the algebraic notation, when by so

doing we could present the subject in a clearer and more satisfactory manner.

Throughout the *Second Part*, which is of a more mixed or practical nature, we have studied to present the whole in a distinct and clear manner, giving in all cases the full and complete work of all examples necessary to illustrate each principle. We would direct the attention in particular to the chapters on Spherical Trigonometry. This subject is generally regarded as intricate and difficult. It is believed the student will find the arrangement and full development of the different cases as here given, nearly as simple and of as easy comprehension as those of Plane Trigonometry.

We might particularize other portions of the work, but will content ourselves by remarking that we have spared no pains to make the whole acceptable to the mathematical student of the present age. How well we have succeeded remains to be shown.

GEO. R. PERKINS.

URICA, September, 1854.

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PART FIRST.

PLANE AND SOLID
GEOMETRY.

INTRODUCTION.

1. ALL bodies occupy, in the *indefinite space* which embraces the material universe, a *determinate*, or *finite place* which we properly call a *space*.

This finite space occupied by the body, has *limits* or *boundaries*, which separate it from the rest of space. These boundaries constitute what we call the *Surface* of the body. Surface being then the separation of a body from the rest of space, belongs as well to the one as the other; and as an infinite number of bodies may exist, in indefinite space, each having its proper surface, it follows that

In space we may conceive of an infinite number of surfaces.

When one surface is cut by another surface, the place of their mutual intersection is called a *Line*. This line belongs to each of the surfaces. Since the intersection of two surfaces gives a line, and any surface may be intersected by an infinite number of other distinct surfaces, it follows that

On any surface whatever, we may conceive an infinite number of lines.

The place of meeting, or of intersection of two lines, is called a *Point*. This point is common to the two lines. Since a point results from the meeting of two lines, and any line may be met by an infinite number of other distinct lines, it follows that

Any line may be regarded as having an infinite number of points.

2. Although we acquire the notion of a point by the consideration of lines, the notion of a line by the consideration of surfaces, and that of the surface by the consideration of a body; that is to say, of *material* things, we must not conclude from this, that points, lines, and surfaces are themselves *material* objects. In virtue of an inherent faculty of the mind, we may readily conceive of a point without the lines which determine it,

of a line without the surfaces of which it is the intersection, of a surface independently of the body, or of the space of which it is the limit; in short, we may conceive of space itself, as absolutely immaterial. It is the result of these different abstractions which we call *a point, a line, a surface, or a space*. It is the same when we speak of *the points of a line, of a surface, of a space; the lines of a surface, &c.*

3. A space, a surface, and a line, may be considered in two distinct ways. We may consider them in regard to their different *forms*, which we name in general their *Figures*. Or, we may consider them in regard to their relative *magnitudes*, which is comprehended under the name of *Extension*.

Extension takes the particular name of *Volume*, of *Area*, or of *Length*, according as it is applied to a space, a surface, or a line. Thus, the length of a line, or its *linear extension*, is the magnitude of this line, estimated or measured in units of a line. In the same way the *area* of a surface, or its *superficial extension*, is the magnitude of this surface, estimated or measured in units of a surface. In short, the *volume* or the *extension of space* is the magnitude of this space, estimated or measured in units of space.

GEOMETRY is the science which treats of these two principal objects: The *properties* of different kinds of *figures*, and the *measure of extension*, considered under the different circumstances, as already noticed.

THE POINT.

4. *A point has neither figure nor extension*; it is this, above all, which distinguishes it from all other objects of Geometry, which are all capable of being *described* and *measured*.

Nevertheless as it is often necessary to consider one or more isolated points, we represent their position by a dot or distinct mark made on a surface with the pen, pencil, or crayon, and we distinguish them from each other, usually, by letters. Thus we say the point A, the point B, the point C, &c.

.A .B .C .D

The position of a point in reference to any other point is determined by its direction and distance from that point. This *distance*, which is the length of the shortest line which can

unite these points, is obviously mutual to the two points; that is, the distance of the point B from the point A is the same as the distance of the point A from the point B.

The *direction* of the point A from the point B is *opposite* to the *direction* of the point B from the point A.

THE STRAIGHT LINE.

5. Of all geometrical lines, the simplest is the *straight line*.

Although the idea of a straight line, is the first to which we are conducted by our experience, and the use of our senses, still it is very difficult to define it. The definition usually given is as follows: *A straight line is the shortest distance between two points.*

A more general definition is as follows: A straight line is an indefinite line, such, that any limited portion whatever, is the shortest distance between the points which fix this limit.

In any line the direction of one of its points, B, from another point, A, as has already been noticed, is opposite the direction of A from B; so that a line has *two different directions exactly opposite*, either of which may be regarded as the direction of the line.

A Straight line is one which has the same direction throughout its whole extent.

A Curved line is one which changes its direction at every point.

Two straight lines are evidently capable of *superposition*, that is, of being placed the one on the other, so as to coincide. Hence, two straight lines coincide throughout their whole extent when they have two points common. Or, in other words, two points determine the position of a straight line. We also infer that two distinct straight lines can intersect or meet each other in only one point.

THE PLANE.

6. The plane surface, or, as usually expressed, the *Plane*, is the simplest of all surfaces. It may be defined as follows: A plane is an indefinite surface, on which we may conceive that through each of its points a straight line may be made exactly to coincide with it throughout its whole extent.

It immediately results from this definition, and the nature of

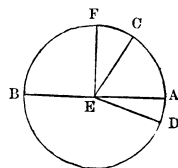
a straight line, That any line, having two of its points common with the plane, lies wholly in this plane. Hence, a straight line cannot be partly in a plane and partly out of it.

When a straight line has only one point in common with a plane, it is said to meet or pierce the plane, and the plane is said to cut the line, and the segments or portions of the line thus separated will be on different sides of this plane.

THE CIRCLE.

7. When a line is not a straight line, or made up of finite portions of straight lines, it is called a *curved line*.

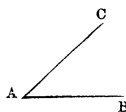
The simplest of all *curved lines* is the *circumference of a circle*, which may be thus defined: *The circumference of a circle* is a plane curve returning into itself, every point of which is equally distant from a certain point in its plane, which point is called the *centre* of the circumference. The portion of the plane limited by this circumference is called a *circle*. Any portion of the circumference of a circle is called an *arc*. When the arc is equal to one-fourth of the circumference it is called a *quadrant*. Each of the straight lines drawn from the centre to the circumference is called a *radius*. Hence, *all radii of the same circle are equal*.



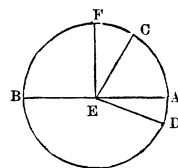
A line passing through the centre and terminating in both directions by the circumference, is called a *diameter*. *All diameters of the same circle are equal*, since each is twice the radius.

THE ANGLE.

8. When two straight lines meet, the opening between them is called a *plane angle*, or simply an *Angle*. The magnitude of the angle does not depend upon the lengths of these lines, but only upon the difference of their directions.



If in any circle we draw two radii, the distance between their extremities which terminate in the circumference, will embrace an arc. When the arc between the two radii is equal to a quadrant, these radii form with each



other an angle, which we call a *right angle*. And the radii are said to be the one *perpendicular* to the other. When the arc between the radii is less than a quadrant, the angle is called *acute*. When the arc is greater than a quadrant, the angle is called *obtuse*.

The magnitude of an angle may be estimated or measured by means of any particular angle, taken as the unit angle. The right angle is generally the angle chosen as the unit angle.

THE RULER AND THE COMPASS.

9. The straight line and the circumference of the circle, which are the only lines treated of in Elementary Geometry, are respectively traced or drawn upon a plane, by the aid of the *Ruler* and of the *Compass*.

These instruments are so simple, and of such general use, as to need no description in this place. With the Ruler we can draw a straight line on a plane from any one point to any other point. With the Compass we can describe on a plane the circumference of a circle having any given point as a centre, and for its radius any given line.

METHODS OF DEMONSTRATION.

10. There are two distinct methods employed in Geometrical demonstration: The *Direct Method*, and the *Indirect Method*.

The most simple process of direct demonstration is the principle of *superposition*, which consists in being able to make two figures exactly coincide, by applying the one upon the other.

The demonstration is also direct when we employ, by a direct course of reasoning, axioms, definitions, and principles already established.

The indirect method, known under the name of *Reducing to an absurdity*, consists in first supposing the proposition not to be true; afterwards, by certain deductions, to draw, from truths already recognized as rigorously exact, a result contradictory to some one of these truths, or to the proposition itself.

We will terminate this subject by noticing two kinds of *false reasoning*, very common with beginners, and against which they should be constantly on their guard.

The first is called, *Reasoning in a circle*.

The second is called, *Begging the question*.

We are said to reason in a circle when, in the demonstration of a proposition, we employ, either implicitly or explicitly, a second proposition, which cannot, itself, be established without the aid of the first.

We are said to beg the question, when, in order to establish a proposition, we employ the proposition itself.

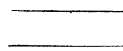
GEOMETRY.

FIRST BOOK.

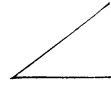
THE PRINCIPLES.

DEFINITIONS.

- I. GEOMETRY is the science of *Position* and *Extension*.
- II. A *Point* has merely position, without any extension.
- III. *Extension* has three dimensions; *Length*, *Breadth*, and *Thickness*.
- IV. A *Line* has only one dimension; length.
- V. A *Surface* has two dimensions; length and breadth.
- VI. A *Solid* has three dimensions; length, breadth, and thickness.
- VII. A *Straight line* is one which has the same direction through its whole extent. In reality a line has two directions, the one exactly opposite the other; either of which may be considered as its direction.
- VIII. A *Broken line* is one which is made up of two or more straight lines.
- IX. A *Curved line* is one which changes its direction at every point.
- X. *Parallel lines* are those which have the same direction.
- XI. An *Angle* is the difference in direction of two straight lines meeting or crossing each other.
- The *Vertex* of the angle is the point where its *sides* meet.
- XII. When one straight line meets or crosses another, so as to make the adjacent angles equal, each of these angles is called a *Right angle*, and the lines are said to be *perpendicular* to each other.



XIII. An *Acute angle* is one which is less than a right angle.



XIV. An *Obtuse angle* is one which is greater than a right angle.



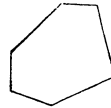
XV. When the sum of two angles is equal to a right angle, they are called *Complementary angles*; each being the complement of the other.

XVI. When the sum of two angles is equal to two right angles, they are called *Supplementary angles*; each being the supplement of the other.

XVII. A *Plane* is a surface which is straight in every direction, or one with which a straight line, joining any two of its points, will coincide.

XVIII. When a surface is not a plane surface it is called a *Curved surface*.

XIX. A *plane figure* is a limited portion of a plane. When it is limited by straight lines, the figure is called a *rectilineal figure*, or a *polygon*; and the limiting lines, taken together, form the *contour* or *perimeter* of the polygon.

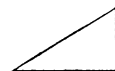


XX. The simplest kind of polygon is one having only three sides, and is called a *triangle*. A polygon of four sides is called a *quadrilateral*; that of five sides is called a *pentagon*; that of six sides is called a *hexagon*; one of seven sides is called a *heptagon*; one of eight sides an *octagon*; one of nine sides a *nonagon*, and so on for figures of a greater number of sides.

XXI. A triangle having the three sides equal, is called an *equilateral triangle*; one having two sides equal, is called an *isosceles triangle*; and one having no two sides equal, is called a *scalene triangle*.



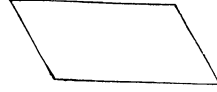
XXII. A triangle having a right angle, is called a *right-angled triangle*. The side opposite the right angle is called the *hypotenuse*.



XXIII. A triangle having its three angles acute, is called an *acute-angled triangle*.

XXIV. A triangle having an obtuse angle, is called an *obtuse-angled triangle*.

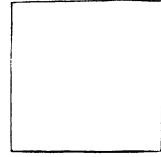
XXV. When the opposite sides of a quadrilateral are parallel, the figure is called a *parallelogram*.



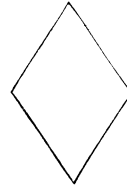
XXVI. When the four angles of a parallelogram are right angles, the figure is called a *rectangle*.



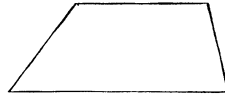
XXVII. When the four sides of a rectangle are equal, the figure is called a *square*.



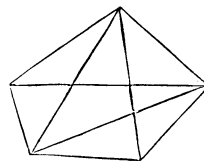
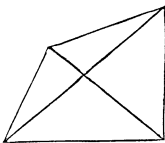
XXVIII. When the four sides of a parallelogram are equal, and the angles not right, the figure is called a *rhombus*.



XXIX. When only two sides of a quadrilateral are parallel, the figure is called a *trapezoid*.



XXX. A *diagonal* of a polygon is a line joining the vertices of two angles, not adjacent.



DEFINITIONS OF TERMS.

I. A *demonstration* is a logical train of reasoning employed to establish an asserted truth.

II. A *proposition* is a statement of a truth to be demonstrated, or of an operation to be performed.

III. An *axiom* is a self-evident proposition, of which the simple mention carries a conviction of its truth.

IV. A *postulate* is a proposition, the truth of which is required to be admitted without demonstration, notwithstanding it does not present the same degree of evidence as in the case of the axiom.

V. A *theorem* is a proposition which requires a demonstration.

VI. A *corollary* is an immediate consequence of one or more propositions. If any new course of reasoning is required to establish it, this reasoning is so simple that it may be supplied without much inconvenience.

VII. A *problem* is a question proposed, which requires a solution.

VIII. A *lemma* is a subsidiary truth, employed for the demonstration of a theorem, or the solution of a problem.

IX. A *scholium* is a remark on one or several preceding propositions, which tends to point out their connection, their use, their restriction, or their extension.

X. An *hypothesis* is a supposition, made in the statement of a proposition, or in the course of its demonstration.

REMARKS IN REFERENCE TO SIGNS, SYMBOLS, ETC.

The signs and symbols which we shall employ, have the same signification in Geometry as in Algebra, to which, for their full explanation, we shall refer the student.

For the doctrine of ratios and proportions, we will also refer the student to the method explained in the Algebra.

There is this difference between geometrical ratios of magnitudes, and ratios of numbers: All numbers are *commensurable*; that is, their ratio can be accurately expressed: but many magnitudes are *incommensurable*; that is, their ratio can be expressed only by approximation; which approximation may, however, be carried to any extent we desire. Such is the ratio of the circumference of a circle to its diameter, the diagonal of a square to its sides, etc. Hence many have deemed the arithmetical method not sufficiently general to apply to geometry. This would be a safe inference, were it necessary in all cases to assign the specific ratio between the two terms compared. But this is not the case. Such ratios themselves may be unknown,

indeterminate, or irrational, and still their equality or inequality may be as completely determined by the arithmetical method as by the more lengthy method of the Greek geometers.

A X I O M S .

I. Things which are equal to the same thing, are equal to each other.

II. When equals are added to equals, the wholes are equal.

III. When equals are taken from equals, the remainders are equal.

IV. When equals are added to unequals, the wholes are unequal.

V. When equals are taken from unequals, the remainders are unequal.

VI. Things which are double of the same or equal things, are equal.

VII. Things which are halves of the same thing, are equal.

VIII. Every whole is equal to all its parts taken together, and greater than any of them.

IX. Things which coincide, or fill the same space, are identical.

X. All right angles are equal to one another.

XI. A straight line is the shortest distance between two points.

XII. Through the same point only one straight line can be drawn parallel to another.

XIII. Only one straight line can be drawn joining two given points.

XIV. Straight lines which are parallel to the same line are parallel to each other.

P O S T U L A T E S .

I. Let it be granted that a straight line may be drawn from any point to any other point.

II. That a terminated straight line may be produced, in either direction, to any extent.

III. That a straight line may be bisected or halved.

IV. That from a point, either within or without a straight line, a perpendicular may be drawn to the line.

V. That a line may be drawn, making any given angle with another line.

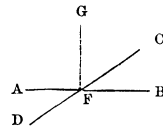
VI. That a line may be drawn from the vertex of an angle, bisecting it.

OF ANGLES.

THEOREM I.

When a straight line meets or crosses another, the adjacent angles are supplements; and the opposite angles are equal.

For, drawing FG perpendicular to AB , we see that the angle AFC exceeds the right angle AFG , by the angle CFG ; while the angle BFC is less than the right angle BFG by the same angle CFG . Hence, the sum of AFC , BFC is equal to two right angles, and they are therefore supplements of each other (D. XVI).*

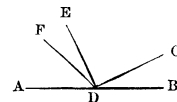


In the same way by drawing a line perpendicular to CD , it may be shown that the angles CFA and AFD are supplements; consequently, $BFC + CFA = CFA + AFD$ (A. I.): from each taking CFA , we have BFC equal to its opposite angle AFD . In a similar manner we have $CFA = DFB$.

Cor. I. If either of the four angles, formed by the intersection of two straight lines, is a right angle, the remaining three angles will each be right, and the two lines will be mutually perpendicular.

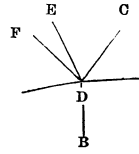
When the angles are not right there will be two equal acute angles, and two equal obtuse angles. The acute angle and the obtuse angle will be supplementary.

Cor. II. All the angles which can be made at any point D , by any number of lines on the same side of AB , are together equal to two right angles.

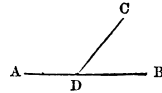


* In the references we shall use the following abbreviations: A. for Axiom, B. for Book, C. for Corollary, D. for Definition, P. for Problem, S. for Scholium, T. for Theorem.

Cor. III. And as all the angles that can be made on the other side of AB are also equal to two right angles; therefore all the angles that can be made around the point D, by any number of lines, are together equal to four right angles.



Cor. IV. If two adjacent angles, ADC and BDC, are supplementary, their exterior sides AD and BD will form one and the same line.

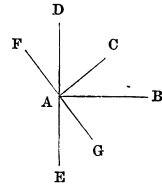


Scholium. Since a straight line has two different directions, exactly opposite each other, it is sometimes considered as making with itself an angle equal to two right angles.

THEOREM II.

If through the vertex of any angle, lines are drawn perpendicular respectively to its sides, they will form a new angle, either equal to the first, or supplementary to it.

Let BAC be the given angle, DE perpendicular to AB, and FG perpendicular to AC.

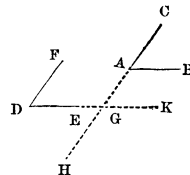


Then we shall have the angle DAF equal to BAC, since each is the complement of CAD (D. XV.). The angle EAG being opposite DAF must also equal BAC (T. I). But the angle DAG, or its opposite angle FAE, is supplementary with DAF, and consequently supplementary with its equal angle BAC.

THEOREM III.

Two angles having their corresponding sides parallel are either equal or supplementary.

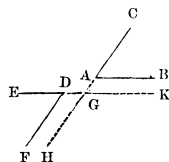
First. When the sides AB and AC, forming the angle at A, have respectively the same directions as the sides DE and DF forming the angle at D.



Produce CA and DE until they cross each other at G. Now, since AB and DE are parallel, they have the same direction in reference to the line CG (D. X.), consequently the angle BAC is equal to KGC. And since AC and DF are parallel they have the same direc-

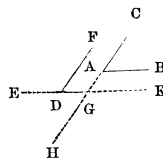
tion in reference to the line DG; consequently, the angle KGC is equal to EDF; therefore (A. I.), the angle BAC is equal to EDF.

Secondly. When the sides AB and AC are respectively in opposite directions to DE and DF.



As before, we have the angle BAC equal to KGC, and the angle EDF equal to EGH. But KGC is equal to its opposite angle EGH (T. I.), consequently, the angle BAC is equal to EDF.

Thirdly. When one of the sides, as AB, has an opposite direction to DE, its corresponding side.



As before, the angle BAC is equal to KGC, and EDF is equal to EGC. But KGC and EGC are supplementary (T. I.), consequently BAC and EDF are supplementary.

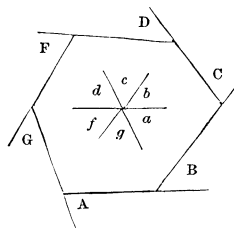
Cor. Two angles having their corresponding sides perpendicular, are either equal or supplementary.

For, if we draw through the vertex of the first angle lines respectively parallel to the sides of the second angle, we should thus form a third angle, which will be equal to the second. But this third angle is either equal to the first or supplementary to it (T. II.). Consequently the second angle is either equal to the first or supplementary to it (A. I.).

THEOREM IV.

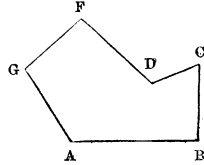
When all the sides of a polygonal figure are produced, in the same direction, the sum of all the exterior angles will be equal to four right angles.

For, if from any point in the same plane, straight lines be drawn parallel respectively to the sides of the figure, the angles contained by the straight lines about that point will be equal to the exterior angles of the figure (T. III.), each to each. Thus the angles a, b, c , etc., are respectively equal to the exterior angles A, B, C, etc.; but the former angles are together equal



to four right angles (T. I., C. III.); therefore all the exterior angles of the figure are together equal to four right angles.

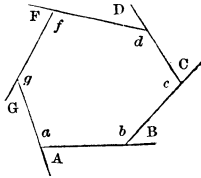
Scholium. This proposition must be restricted to the case in which the polygon is convex. A convex polygon may be defined to be one, such that no side, by being produced in either direction, can divide the polygon. The polygon ABCDFG is not convex, since it may be divided by producing either of the sides CD or FD. This polygon is said to have a *re-entering* angle at D.



THEOREM V.

In any convex polygon, the sum of all the interior angles, taken together, is equal to twice as many right angles as the polygon has sides, wanting four right angles.

Let ABCDFG be a convex polygon. Conceive the sides to be produced all in the same direction, forming exterior angles, which we will denote by the capital letters A, B, C, etc., while their corresponding interior angles are denoted by the small letters a , b , c , etc. Now any exterior angle, together with its adjacent interior angle, as $A + a$, is equal to two right angles (T. I.), therefore the sum of all the interior angles, together with all the exterior angles, is equal to twice as many right angles as the polygon has sides; but the sum of all the exterior angles is equal to four right angles (T. IV.); therefore the sum of all the interior angles is equal to twice as many right angles as the polygon has sides, wanting four right angles.



Cor. I. In any triangle, the sum of the three angles is equal to two right angles.

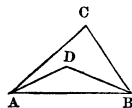
Cor. II. If one angle of a triangle is a right angle, as in the case of a right-angled triangle (D. XXII.), each of the other angles must be acute, and they are complementary, since their sum must equal a right angle. Each of the angles of a triangle may be acute, but only one of the angles can be obtuse.

Cor. III. If two triangles have two angles of the one respectively equal to two angles of the other, their third angles will be equal, and the triangles will be mutually equiangular.

Cor. IV. In any quadrilateral, the sum of the four interior angles is equal to four right angles.

Cor. V. If two angles of a quadrilateral are right, the other two angles will be supplementary.

Cor. VI. If from a point D within a triangle ABC , we draw two lines to the extremities of one of the sides AB , the angle ADB formed by these lines will be greater than the angle ACB , opposite this side.

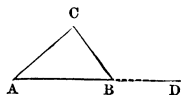


For, the sum of the two angles DAB , DBA is obviously less than the sum of the two angles CAB , CBA . And since the sum of the three angles of every triangle is equal to two right angles, it follows that the remaining angle ADB , of the triangle ADB , must be greater than the remaining angle ACB of the triangle ACB .

THEOREM VI.

The exterior angle, formed by producing one of the sides of a triangle, is equal to the sum of the two interior and opposite angles of the triangle.

In the triangle ABC , if the side AB be produced to D , the exterior angle CBD will be equal to the sum of the two angles at A and at C .



For, the angle CBD , together with its adjacent angle, CBA , is equal to two right angles (T. I.), and the sum of A and C , together with the same angle CBA , is also equal to two right angles (T. V., C. I.); hence, the exterior angle is equal to the sum of the two interior and opposite angles.

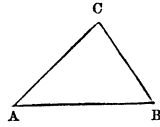
Cor. The exterior angle is greater than either of the interior and opposite angles.

OF THE SIDES OF TRIANGLES.

THEOREM VII.

Either side of any triangle is less than the sum of the other two sides, and greater than their difference.

First. The straight line AB is the shortest distance between A and B (A. XI.), and therefore shorter than the broken line $AC + CB$. The same reasoning applies to each of the sides. Hence, we have $AB < AC + CB$; $AC < AB + CB$; $BC < AB + AC$.

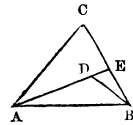


Secondly. Since $AC < AB + CB$, we have $AC - CB < AB$ (A. V.). That is, $AB > AC - CB$. In a similar manner we deduce $AC > AB - CB$; $CB > AB - AC$.

THEOREM VIII.

If from any point within a triangle two lines be drawn to the extremities of either side, the sum of these two lines will be less than the sum of the other two sides.

From the point D , suppose the lines DA and DB to be drawn to the extremities of the side AB ; then will $DA + DB < AC + BC$.



For, producing AD until it meets BC at E , we have (T. VII.)

$$\begin{aligned} DB &< DE + EB, \text{ consequently (A. IV.),} \\ AD + DB &< AD + DE + EB, \text{ that is,} \\ AD + DB &< AE + EB. \end{aligned}$$

Again, we have

$$\begin{aligned} AE &< AC + CE, \text{ consequently,} \\ AE + EB &< AC + CE + EB, \text{ that is,} \\ AE + EB &< AC + CB. \end{aligned}$$

Comparing these conditions, we have

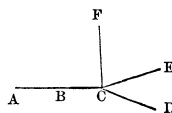
$$AD + DB < AC + CB.$$

OF PERPENDICULAR AND OBLIQUE LINES.

THEOREM IX.

If two straight lines have two points common they will coincide throughout their whole extent.

If A and B are points common to two straight lines, they will coincide between A and B (A. XIII.).

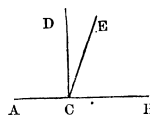


If possible, we will suppose that when produced, they begin to separate at C, the one taking the direction CD, and the other the direction CE. Draw CF perpendicular to AC; then since the lines ACE and ACD are each straight, we have the angle FCD and FCE equal, each being equal to a right angle (T. I., C. I.); that is, the whole is equal to one of its parts, which is impossible (A. VIII.). It is therefore absurd to suppose these lines can separate when produced. Hence, if two straight lines have two points common they will coincide throughout their whole extent.

THEOREM X.

Through a given point in a straight line only one perpendicular can be drawn to this line.

If there could be two perpendiculars, as CD and CE, the angles BCD and BCE would be equal, each being a right angle; that is, the whole would be equal to its part, which is impossible (A. VIII.). Hence, it is absurd to suppose that more than one perpendicular can be drawn to a given line through any one of its points.

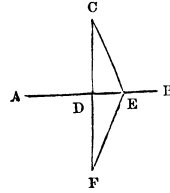


THEOREM XI.

From a given point without a straight line, only one perpendicular can be drawn.

Let C be the point, and AB the given line. If possible, suppose we can draw the two perpendiculars CD and CE. Revolve

the figure CDE about DE as a hinge, until it returns into its primitive plane on the opposite side of AB, having the position FDE. Since each of the angles FDE and CDE is right, CDF is a straight line (T. I., C. IV.). By the same reasoning we have CEF a straight line. That is, we have two straight lines joining the points C and F, which is impossible (A. XIII.). Hence, from a point without a straight line only one perpendicular can be drawn to this line.



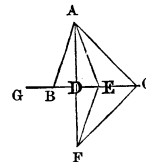
THEOREM XII.

If from a point without a line, a perpendicular be drawn, and several oblique lines :

- I. *The perpendicular will be shorter than any oblique line.*
- II. *Any two oblique lines which terminate at equal distances from the foot of the perpendicular, will be equal.*
- III. *Of two oblique lines terminating at unequal distances from the foot of the perpendicular, the one at the greater distance will be the longer.*

Let A be the given point, BC the given line, AD perpendicular, and AB, AE, and AC oblique lines.

First. As in the last Theorem, suppose ADC to revolve about BC into the position FDC, on the opposite side of BC in its primitive plane. We have $FD=AD$; $FE=AE$; and $FC=AC$. But since each of the angles FDE and ADE is a right angle, ADF is a straight line (T. I., C. IV.), and therefore shorter than the broken line $AE+EF$; hence, AD, the half of AF, is less than AE, the half of $AE+EF$. That is, the perpendicular is shorter than any oblique line.



Secondly. If we suppose the figure ADB to revolve about AD, the point B will coincide with E, since DB is equal to DE, and the angle ADB is equal to ADE, each being a right angle. Hence, the oblique line AB will coincide with AE. That is, two oblique lines which terminate at equal distances from the foot of the perpendicular are equal.

Thirdly. Returning to the figure as first revolved, we have,

since E is a point within the triangle ACF, $AC + CF > AE + EF$ (T. VIII.); hence, AC, the half of $AC + CF$, is greater than AE, the half of $AE + EF$. That is, the oblique line terminating farther from the foot of the perpendicular is the longer.

Cor. I. The perpendicular measures the shortest distance from a point to a line.

Cor. II. From the same point without a straight line, only two equal oblique lines can be drawn, one on each side of the perpendicular.

THEOREM XIII.

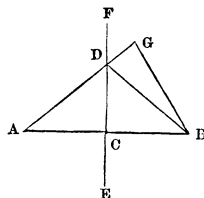
If through the middle point of a straight line, a perpendicular be drawn :

I. *Any point in this perpendicular will be equally distant from the extremities of this line.*

II. *Any point without the perpendicular will be unequally distant from the extremities of this line.*

Let AB be the given line, DE a perpendicular through C, its middle point.

First. Let D be any point in this perpendicular. Drawing DA and DB, we know these lines to be equal, since they terminate at equal distances from C, the foot of the perpendicular DC (T. XII.). That is, D is equidistant from A and B. In the same manner we may show that any other point of this perpendicular, as E or F, is equidistant from the extremities of the given line.



Secondly. Suppose G to be a point without the perpendicular. If we draw GA and GB, one of these lines must cut the perpendicular. We will suppose GA to cut it at D. Draw DB, then GB will be less than $GD + DB$ (T. VII.). But, we already have $DB = DA$, hence GB is less than $GD + DA$, or less than GA. That is, the point G is unequally distant from the extremities of the given line.

In the same manner we can show that any other point without the perpendicular is unequally distant from the extremities of this line.

Scholium. The most distant extremity A of this line will be on the side of the perpendicular opposite the point G.

Cor. I. Any point equidistant from the extremities of a straight line is situated in the perpendicular which bisects this line.

Cor. II. If a straight line have any two of its points equidistant from the extremities of a second line, it will be perpendicular to the second line and bisect it.

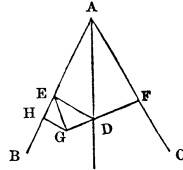
THEOREM XIV.

If a line be drawn bisecting a given angle, that is, dividing it into two equal angles :

I. Any point in this bisecting line will be equidistant from the sides of the angle.

II. Any point without this bisecting line will be unequally distant from the sides of the angle.

First. Let the given angle BAC be bisected by the line AD , then will any point as D in this line be equidistant from the lines AB and AC .



The distance of D from these lines will be measured by the perpendiculars DE and DF (T. XII., C. I.).

If we conceive the figure folded about the line AD , so that the portion on the left of AD may be superposed upon the portion on the right, the line AB will then coincide with AC , since the angle $BAD = CAD$. Consequently the perpendiculars DE and DF will coincide (T. XI.), and are therefore equal.

Secondly. Suppose G to be a point without the bisecting line. If we draw the perpendiculars GF and GH , one of these must cut the bisecting line. Let GF cut it at D . Draw DE perpendicular to AB , and join G and E : then will GH be less than GE (T. XII.). But GE is less than $GD + DE$ (T. VII.); therefore, GH is less than $GD + DE$, or less than $GD + DF$, since by the first case of this Theorem $DF = DE$. Hence, finally, GH is less than GF .

Cor. I. Any point equidistant from the sides of a given angle is situated on the line bisecting this angle.

Cor. II. If a straight line have two of its points equidistant from the sides of a given angle, it will bisect this angle.

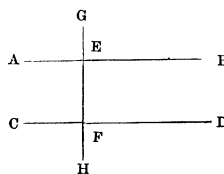
OF PARALLEL LINES.

THEOREM XV.

When two lines are perpendicular to the same line, they are parallel.

If the lines AB and CD are perpendicular to GH they will be parallel.

For, since they are perpendicular to GH, we have the angles BEG and DFE equal, each being a right angle; hence, these lines have the same direction, and are therefore parallel (D. X.).



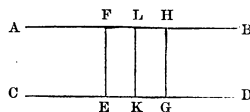
Cor. A line which is perpendicular to one of two parallels, is also perpendicular to the other.

THEOREM XVI.

Two parallel lines are throughout their whole extent equally distant.

Suppose AB and CD to be parallel.

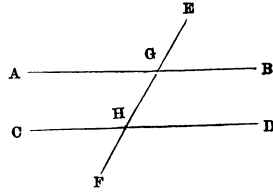
Through any two points, as E and G, of the line CD, draw EF and GH perpendicular to CD, they will also be perpendicular to AB (T. XV.), and they will measure the distance of E and G respectively from the line AB (T. XII., C. I.).



We are now to prove these lines equal. Through K, the middle of EG, draw KL perpendicular to CD, and it will also be perpendicular to AB. Now, suppose the portion of the figure on the left of KL to revolve about KL as a hinge, until it returns into its primitive plane, on the right of KL. The angles at K and L being right, KE will take the direction of KG, and LF the direction of LH, and E will coincide with G, since $KE = KG$, and since EF and GH are each perpendicular to CD, they must coincide, which proves them equal.

Cor. Two parallels cannot meet, however far they are produced.

Scholium. When two parallels are cut by a third line, the angles formed receive particular names.



WHEN TAKEN SEPARATELY :

1. The four angles AGF, CHE, BGF, and DHE, situated within the parallels, are called *interior angles*.
2. The four angles AGE, CHF, BGE, and DHF, situated without the parallels, are called *exterior angles*.

WHEN COMPARED TWO AND TWO :

1. The two angles AGF and CHE, as well as BGF and DHE, are called *interior angles on the same side*; that is, on the same side of the *secant* or cutting-line.
2. The two angles AGE and CHF, as well as BGE and DHF, are called *exterior angles on the same side*.
3. The two angles AGF and DHE, as well as CHE and BGH, being interior angles on different sides of the secant, are called *alternate interior angles*.
4. The two angles AGE and DHF, as well as CHF and BGE, are called *alternate exterior angles*.

We have noticed two pairs of each of the four distinct kinds of angles.

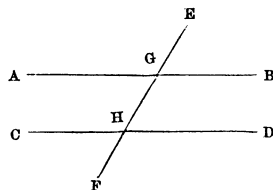
5. There are still four pairs of angles, referred to as *Corresponding angles*, namely: AGE and CHE, AGF and CHF, BGE and DHE, BGH and DHF. These last angles, which establish the direction of the lines, are sometimes called *opposite exterior* and *interior angles*.

THEOREM XVII.

If two straight lines are cut by a third line, making the sum of the two interior angles on the same side equal to two right angles, the two lines will be parallel.

If the two lines AB and CD are cut by the line EF, making $BGH + DHG = 2$ right angles. these lines will be parallel.

For we have $BGH + BGE = 2$ right angles (T. I.), therefore $BGH + DHG = BGH + BGE$; taking BGH from each we have $DHG = BGE$, hence the lines AB and CD have the same direction, and are consequently parallel (D. X.).



Cor. If two straight lines are cut by a third line, they will be parallel:

I. *When the sum of the exterior angles on the same side is equal to two right angles.*

II. *When the alternate interior angles are equal.*

III. *When the alternate exterior angles are equal.*

IV. *When the corresponding angles are equal.*

First. Suppose $EGB + FHD = 2$ right angles.

We have $EGB = AGH$, and $FHD = CHG$ (T. I.), hence $AGH + CHG = 2$ right angles, which agrees with the Theorem itself, hence AB and CD are parallel.

Secondly. Suppose $AGH = GHD$.

We have $AGH = EGB$ (T. I.), consequently $EGB = GHD$; that is, the lines AB and CD have the same direction, and are therefore parallel.

Thirdly. Suppose $EGB = CHF$.

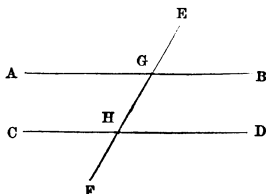
We have $CHF = GHD$, consequently $EGB = GHD$, and the lines have the same direction, and are therefore parallel.

Fourthly. Since the corresponding angles are equal, the lines have the same direction, and are therefore parallel.

THEOREM XVIII.

If two parallels are cut by a third line, the sum of the two interior angles on the same side is equal to two right angles.

Since the lines are parallel they have the same direction, and their corresponding angles must be equal. We therefore have $EGB = GHD$; to each add BGH , and we shall have $EGB + BGH = BGH + GHD$.



But $EGB + BGH = 2$ right angles (T. I.), consequently $BGH + GHD = 2$ right angles.

Cor. It follows immediately from D. X., in connection with T. I., C. I., that if either one of the eight angles formed by the intersection of a straight line with two parallels is a right angle, the other seven will also be right. When these angles are not right, there will be four equal acute angles, and four equal obtuse angles. These angles will be supplementary. Any two of the acute angles, or any two of the obtuse angles taken together, constitute a pair of corresponding angles, and fix the direction of the parallels. Hence it readily follows, that—

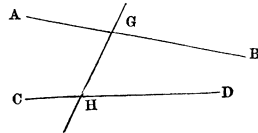
If two parallels are cut by a third line, we shall have :

- I. *The sum of the exterior angles on the same side equal two right angles.*
- II. *The alternate interior angles equal.*
- III. *The alternate exterior angles equal.*
- IV. *The corresponding angles equal.*

THEOREM XIX.

If two straight lines are cut by a third line, and the sum of the interior angles on the same side is not equal to two right angles, the two lines will meet, if sufficiently produced.

If we suppose the angle $AGH + CHG > 2$ right angles, then, obviously, will the angle $BGH + DHG < 2$ right angles, since these angles are respectively supplementary angles. This being



supposed, these lines will meet. For if they do not meet they are parallel; but they are not parallel, since the sum of the interior angles on the same side is not equal to two right angles. Hence these lines will meet if produced sufficiently far.

Scholium. It is evident they will meet in the direction of B and D, on that side of the secant line which has the sum of the interior angles less than two right angles.

OF TRIANGLES.

THEOREM XX.

If two triangles have two sides and the included angle of the one, equal to the two sides and the included angle of the other, the triangles will be identical, or equal in all respects.

In the two triangles ABC, DFG, if the side CA be equal to the side DG, and the side CB equal to the side GF, and the angle C equal to the angle G, then will the two triangles be identical, or equal in all respects.



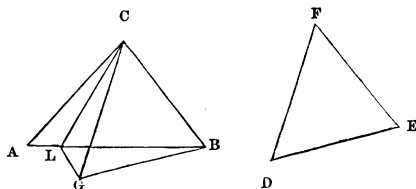
For, conceive the triangle ABC to be placed upon the triangle DFG in such a manner that the point C may coincide with the point G, and the side CA with the equal side GD. Then, since the angle G is equal to the angle C, the side CB will take the direction of the side GF. Also CB being equal to GF, the point B will coincide with the point F; consequently the side AB will coincide with DF. Therefore the two triangles are identical, and have all their other corresponding parts equal (A. IX.), namely, the side AB equal to the side DF, the angle A equal to the angle D, and the angle B equal to the angle F.

Cor. Two right-angled triangles are equal, when the two sides containing the right angle of the one are respectively equal to the two sides containing the right angle of the other.

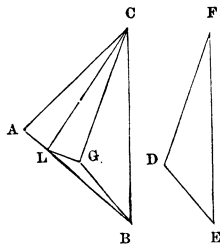
THEOREM XXI.

If two triangles have two sides of the one respectively equal to two sides of the other, and the included angle of the first greater than the included angle of the second, the third side of the first will be greater than the third side of the second.

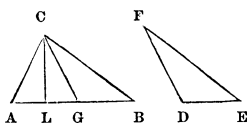
In the two triangles ABC and DEF , suppose CA and CB of the first to be respectively equal to FD and FE of the second, and the included angle $ACB > DFE$, then will $AB > DE$.



Apply the triangle DEF upon the triangle ABC , making EF coincide with BC , so that the triangle may, in its new position, be represented by GBC . Draw CL bisecting the angle ACG (Post. VI.), and meeting AB at L , draw GL . The two triangles ACL , GCL , are equal (T. XX.), and $LG = LA$. Therefore, $AB = AL + LB = LG + LB$, which is greater than BG (T. VII.). That is, $AB > DE$.



Although the triangle DEF , when applied to ABC , may have three different positions, as here represented, still our demonstration is alike applicable to either case.



Cor. Conversely. If two triangles have two sides of the one respectively equal to two sides of the other, and the third side of the first greater than the third side of the second, the included angle of the first will be greater than the included angle of the second.

For, if the included angle of the first is not greater than the included angle of the second, it must be either equal or less. If it were equal, the third sides would be equal (T. XX.). If it were less, the third side of the first would be less than the third side of the second, by the Theorem itself. Both these results are contrary to the hypothesis. Consequently, the included angle of the first is greater than the included angle of the second.

THEOREM XXII.

If two triangles have two angles and the interjacent side of the one equal to two angles and the interjacent side of the other, the triangles will be identical, or equal in all respects.

In the two triangles ABC , DFG , if the angle A is equal to the angle D , the angle B equal to the angle F , and the side AB equal to the side DF , then will the triangles be identical, or equal in all respects.



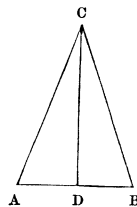
For, conceive the triangle ABC to be placed on the triangle DFG , in such a manner that the side AB may coincide with the equal side DF . Then, since the angle D is equal to the angle A , the side AC will take the direction of the side DG ; also, since the angle F is equal to the angle B , the side BC will take the direction of the side FG ; consequently the point C must coincide with the point G . Therefore the two triangles are identical (A. IX.), having the two sides AC and BC respectively equal to DG and FG , and the remaining angle C equal to the remaining angle G .

THEOREM XXIII.

In an isosceles triangle, the angles at the base are equal; or, if a triangle have two sides equal, the angles opposite those sides will be equal.

If the triangle ABC have the side AC equal to the side BC , then will the angle B be equal to the angle A .

For, conceive the angle C to be bisected, or divided into two equal parts, by the line CD , making the angle ACD equal to the angle BCD . Then the two triangles ADC and BDC will be identical, or equal in all respects (T. XX.), and consequently the angle B is equal to the angle A .



Cor. I. The line which bisects the vertical angle of an isosceles triangle, bisects the base perpendicularly, and divides the triangle itself into two equal parts.

Cor. II. An equilateral triangle is equi-angular, that is, has all its angles equal.

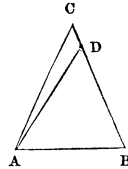
THEOREM XXIV.

If a triangle have two angles equal, the sides opposite those angles will be equal, and the triangle will be isosceles.

In the triangle ABC, if the angles CAB, CBA, are equal, the opposite sides BC, AC, will be equal.

For, if they are not equal, suppose $BC > AC$, and take $BD = AC$. Then in the two triangles BAD and ABC we have the side $BD = AC$ by supposition, the side AB common, and the included angle ABD of the first triangle equal to BAC, the included angle of the second triangle. Hence (T. XX.), the triangle BAD is equal to ABC, that is, a part is equal to the whole, which is impossible (A. VIII.).

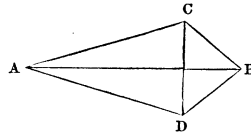
There is, therefore, no inequality of the sides CB and CA, that is, they are equal, and the triangle is isosceles.



THEOREM XXV.

When two triangles have the three sides of the one respectively equal to the three sides of the other, the triangles will be identical, and equal in all respects.

Let the two triangles ABC, ABD, have their sides respectively equal, namely, AB equal to AB, AC equal to AD, and BC equal to BD, then will these triangles be identical.



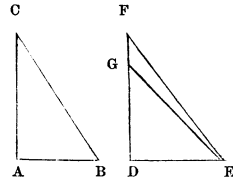
For, conceive the two triangles to be joined together by their longest equal sides, and draw the line CD; then in the triangle ACD, since AC is equal to AD, we have the angle ACD equal to the angle ADC (T. XXIII.). In like manner, in the triangle BCD, since BC is equal to BD, we have the angle BCD equal to the angle BDC. Hence the angle ACB, which is the sum of ACD and BCD, is equal to the angle ADB, which is the sum of ADC and BDC. Since, then, in the triangle ACB, we have the two sides AC and BC, and their included angle ACB, equal respectively to the two sides AD and BD, and their included angle ADB, of the triangle ADB, it therefore follows that these triangles are identical (T. XX.).

THEOREM XXVI.

Two right-angled triangles are equal, when the hypotenuse and a side of the one, are respectively equal to the hypotenuse and a side of the other.

Suppose $BC = EF$ and $AB = DE$, then will the two right-angled triangles, ABC and DEF , be equal.

For, if we apply the triangle ABC upon the triangle DEF , so that AB may coincide with its equal DE , the side AC will take the direction of DF , since the angle at A is equal to the angle at D , each being a right angle. If, now, BC coincides with EF , there will be a complete coincidence, and the two triangles will be equal. If possible, suppose BC to take the position EG , so that the triangle ABC may be represented by DEG ; then, since the hypotenuses of the two triangles are equal, we shall have $EG = EF$, that is, two oblique lines are drawn from E , terminating at unequal distances from D , the foot of the perpendicular ED , which is impossible (T. XII.). It is, therefore, absurd to suppose BC to take any position different from EF . Hence, the triangles coincide throughout, and are in all respects equal.



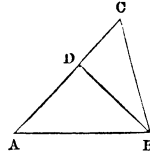
Cor. Two right-angled triangles will also be equal, when the hypotenuse and an acute angle of the one are respectively equal to the hypotenuse and an acute angle of the other. For, since the sum of the two acute angles of any right-angled triangle is equal to a right angle (T. V., C. II.), it follows that the remaining acute angles of the two triangles will be equal. Hence, the two triangles will have two angles and the interjacent side of the one equal, respectively, to two angles and the interjacent side of the other. Consequently, they will be equal (T. XXII.).

THEOREM XXVII.

The greater side of every triangle is opposite the greater angle, and, conversely, the greater angle is opposite the greater side.

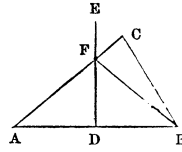
First. Suppose, in the triangle ABC , the angle $CBA > CAB$, then will the side $CA > CB$.

For, draw BD , making the angle $DBA = DAB$, and DA will equal DB (T. XXIV.). To each adding DC , we have $DA + DC = DB + DC$, but $DB + DC$ is greater than CB (T. VII.); hence, $DA + DC$, or its equal CA , is greater than CB .



Secondly. Suppose, in the triangle ABC , the side $CA > CB$, then will the angle $CBA > CAB$.

For, through the middle point of the side AB draw the perpendicular DE . Now, since $CA > CB$, the point C must be on the side of DE opposite A (T. XIII., S.); consequently, the perpendicular must cut CA at some point, as F . Draw FB , and we shall have $FB = FA$ (T. XIII.), and angle $FBA = FAB$ (T. XXIII.); consequently, $FBA + FBC$, which equals CBA , is greater than CAB .

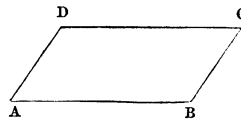


OF QUADRILATERALS.

THEOREM XXVIII.

If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

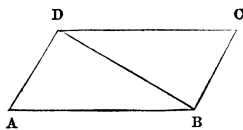
We have the angles $A + B + C + D = 4$ right angles (T. V., C. IV.), true for all quadrilaterals. Now, if we suppose $A = C$, and $B = D$, we shall find $2A + 2D = 4$ right angles, or $A + D = 2$ right angles, hence AB and DC are parallel (T. XVII.). In a similar manner, we find $2A + 2B = 4$ right angles, or $A + B = 2$ right angles, and, as before, AD and BC are parallel. Consequently the figure is a parallelogram (D. XXV.).



THEOREM XXIX.

If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

If we suppose $AB = DC$, and $AD = BC$, and draw the diagonal BD , we shall have the three sides of the triangle ABD respectively equal to the three sides of the triangle CDB , consequently they are equal, (T. XXV.), and the angle $ABD = CDB$, which are alternate interior angles in reference to the sides AB and DC , hence AB and DC are parallel (T. XVII., C.).

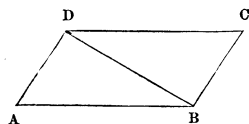


The equality of the two triangles also gives the angle $ADB = CBD$, which are alternate interior angles in reference to the sides AD and BC , hence these sides are parallel. Consequently the figure is a parallelogram (D. XXV.).

THEOREM XXX.

If two opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

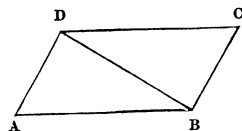
In the quadrilateral $ABCD$, suppose AB equal and parallel to DC , then will the figure be a parallelogram. For, drawing BD , the two triangles ABD , CDB , have the common side BD , and AB of the first equal to CD of the second; also, the included angles ABD , CDB , equal, being alternate angles with reference to the parallels AB , DC . Hence these triangles are equal (T. XX.), and the angle ADB is equal to CBD , but these are alternate angles with reference to AD and BC , consequently AD and BC are parallel, and the figure is a parallelogram.



THEOREM XXXI.

The opposite angles of a parallelogram are equal; so also are the opposite sides equal.

Since the figure is a parallelogram, we have AB and DC parallel, also AD and BC parallel; hence, if we draw the diagonal BD , we shall have the angle $ABD = CDB$, being alternate interior angles in reference to the parallels AB and DC (T. XVIII., C.). For a similar reason



the angle $ADB = CBD$. Therefore, the two triangles ABD and CDB have a common side BD , and the adjacent angles of the one equal respectively to the adjacent angles of the other, consequently they are equal (T. XXII.). Hence the angle $A = C$, and $ABD + CBD = CDB + ADB$, or $ABC = CDA$. That is, the opposite angles of a parallelogram are equal.

The equality of the triangles also gives $AB = DC$, and $AD = BC$; that is, the opposite sides of a parallelogram are also equal.

Cor. I. The diagonal of a parallelogram divides it into two equal parts.

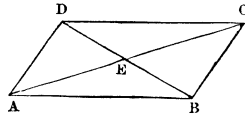
Cor. II. Two parallels included between two other parallels are equal.

Cor. III. If one angle of a parallelogram is right, the other three will be right also.

THEOREM XXXII.

The diagonals of a parallelogram mutually bisect each other, that is, they divide each other into halves.

In the two triangles EAB and ECD , we have the angle EAB of the first equal to ECD of the second, being alternate angles in reference to the parallels AB and DC . For a like reason the angle EBA is equal to EDC . We also have the interjacent sides AB and DC equal, consequently these triangles are equal (T. XXII.), and $AE = EC$, $BE = ED$.



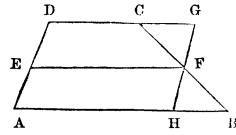
Cor. If the four sides of the parallelogram are equal, as in the case of a *rhombus*, we have $AB = AD$, and the two triangles AEB and AED will have the three sides of the one equal to the three sides of the other respectively, consequently they will be equal (T. XXV.), and the angle $AEB = AED$, that is, in a rhombus the diagonals bisect each other at right angles.

Scholium. The point E , where the diagonals intersect, is called the centre of the parallelogram.

THEOREM XXXIII.

The straight line joining the middle points of the oblique sides of a trapezoid, will be parallel to the other sides, and equal to half their sum.

Suppose EF to join the middle points E and F, of the oblique sides AD and BC of the trapezoid ABCD. Through F draw GH parallel to AD, and meeting DC produced. In the two triangles, FBH and FCG, we have the side FB = FC, the angle FBH equal to its alternate angle FCG, and the angle BFH equal its opposite angle CFG, hence these triangles have two angles and the interjacent side of the one, respectively equal to the two angles and the interjacent sides of the other, and are therefore equal (T. XXII.). Consequently, FG = FH; that is, FG equals half of HG. But HG is equal to AD, since the figure AHGD is a parallelogram. Therefore, FG is half of AD, but ED is also half of AD, hence FG and ED are equal and parallel, and the figure EFGD is a parallelogram, and EF is parallel to DC, and also to AB, since DC and AB are parallel.



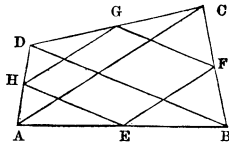
Again, since $EF = DG = AH$; and $CG = HB$, by reason of the equality of the two triangles FGC and FHB, we have EF as much greater than DC as it is less than AB. Hence, EF is one half of the sum of DC and AB.

Cor. If we suppose the side DC of the trapezoid to be reduced to a point, the trapezoid will then change to a triangle. Hence, the straight line joining the middle points of any two sides of a triangle, will be parallel to the third side, and equal to half of it.

THEOREM XXXIV.

The four lines joining the middle points of the adjacent sides of a quadrilateral form a parallelogram.

If we draw the diagonals of the quadrilateral, we shall have EF and HG, each parallel to AC, and equal to half of it (T. XXXIII., C.); consequently they are equal and parallel. For the same reason EH and FG are each parallel to BD, and equal to half of it. Therefore the figure EFGH is a parallelogram.



Cor. The two lines joining the middle points of the opposite sides of a quadrilateral mutually bisect each other, since they are the diagonals of a parallelogram.

ADDITIONAL THEOREMS OF TRIANGLES.

THEOREM XXXV.

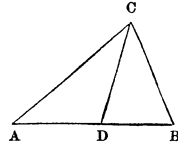
In any triangle if a line be drawn from either angle to the middle of the opposite side:

I. *When this line is equal to the half side, the angle will be a right angle.*

II. *When this line is greater than the half side, the angle will be acute.*

III. *When this line is less than the half side, the angle will be obtuse.*

In the triangle ABC, suppose the line CD to be drawn from the angle C to the middle of the side AB.



First. When $CD = AD = DB$, we have, angle $A = DCA$, angle $B = DCB$ (T. XXIII.), hence $A + B = DCA + DCB$; that is, the angle ACB is one half the sum of the three angles of the triangle, and is therefore a right angle (T. V., C. I.).

Secondly. When $CD > AD$ or DB , we have angle $A > DCA$, angle $B > DCB$ (T. XXVII.), hence $A + B > DCA + DCB$; that is, the angle ACB is less than half the sum of the three angles of the triangle, and it is therefore an acute angle.

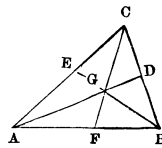
Thirdly. When $CD < AD$ or DB , we have angle $A < DCA$, angle $B < DCB$ (T. XXVII.); hence, $A + B < DCA + DCB$; that is, the angle ACB is greater than half the sum of the three angles of the triangle, and it is therefore an obtuse angle.

Scholium. This Theorem affords a very simple method of determining the kind of angle in any given triangle.

THEOREM XXXVI.

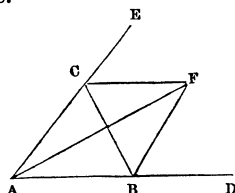
The three lines bisecting the three angles of a triangle, intersect each other in the same point.

Every point in the bisecting line AD is equally distant from AB and AC (T. XIV.). For the same reason every point of the bisecting line BE is equally distant from AB and BC. Hence, the point G, where these two lines intersect, is equally



distant from AC and BC; consequently, it is in the line CF which bisects the angle ACB (T. XIV., C. I.). That is, this point is common to the three bisecting lines.

Cor. If the sides AB and AC are produced, and the exterior angles are bisected by the lines BF and CF, the point F, where they intersect, will be on the line AF, which bisects the angle at A.

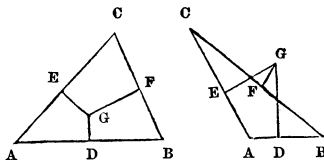


Scholium. This common point, equidistant from the three sides of a triangle, is always within the triangle.

THEOREM XXXVII.

The three perpendiculars bisecting the three sides of a triangle intersect each other in the same point.

Every point in the perpendicular DG is equidistant from A and B (T. XIII.). For the same reason every point in the perpendicular EG is equidistant from A and C. Hence, the point G, where these perpendiculars intersect, is equidistant from B and C; consequently, it is in the perpendicular which bisects the side BC (T. XIII., C. I.). That is, this point is common to the three perpendiculars.

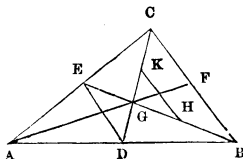


Scholium. This common point, equidistant from the three angles of a triangle, may be either within or without the triangle; or, in the case of a right-angled triangle, it will evidently be at the middle point of the hypotenuse.

THEOREM XXXVIII.

The three lines passing through the three angles of a triangle, and bisecting the opposite sides, will intersect each other in the same point.

We will first consider the two lines CD, BE, which intersect at G. Draw HK joining the middle points of GB, GC, and it will be parallel to BC and equal to half of it (T. XXXIII., C.). For the



same reason DE , which joins the middle points of AB , AC , will be parallel to BC and equal to half of it; consequently, HK and DE are equal and parallel. Now in the two triangles GHK , GED , we have the angle GHK equal to its alternate angle GED , and GKH equal to its alternate angle GDE , also the side $HK = DE$; hence, these triangles are equal (T. XXII.), and $GH = GE$, but GH was taken one half of BG , therefore we have EG one third of EB . For the same reason, DG is one third of DC .

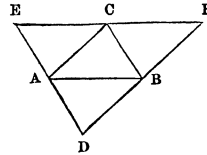
If we now consider the two lines CD , AF , we can, by the same kind of demonstration already employed, show that the point where they intersect is one third of FA , measured from F on FA ; and one third of DC , measured from D on DC , as before. Hence, the three lines intersect at the same point.

Scholium. This common point, situated at one third the distance of the middle points of the sides of a triangle from their opposite angles, is always within the triangle.

THEOREM XXXIX.

The three lines passing through the three angles of a triangle, perpendicular to the opposite sides, will intersect each other in the same point.

Let ABC be the given triangle. Through the vertices draw lines parallel to the opposite sides, thus forming a second triangle DEF .



Then will the figures $ABCE$, $ABFC$, and $ADBC$ be parallelograms, and we have AE and AD each equal to BC ; hence, A is the middle point of ED . For a similar reason, B is the middle point of DF , and C the middle point of EF . From this we see that lines drawn from A , B , C , perpendicularly to BC , AC , AB , will be the same as the three perpendiculars bisecting the sides of the triangle DEF , which perpendiculars we already know must intersect each other in the same point (T. XXXVII.).

Hence, three lines passing through the three angles of a triangle perpendicularly to the opposite sides, will intersect each other in the same point.

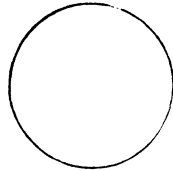
Scholium. This common point may be either within or without the triangle (T. XXXVII.).

SECOND BOOK.

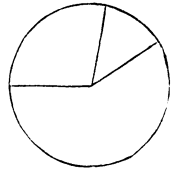
THE CIRCLE, AND ITS COMBINATION WITH THE STRAIGHT LINE.

DEFINITIONS.

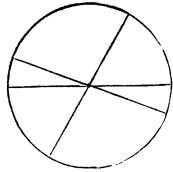
I. The *circumference* of a circle is a curved line, all the points of which are equally distant from a point within called the *centre*. The *circle* is the space bounded by the circumference.



II. Any straight line drawn from the centre to the circumference is called a *radius*. Hence, all radii of the same circle are equal.

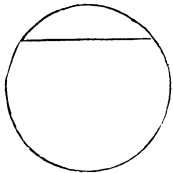


A line passing through the centre and terminating in both directions by the circumference, is called a *diameter*. Hence, all diameters of the same circle are equal, each being made up of two radii.



III. Any portion of the circumference of a circle is called an *arc*. One fourth of the entire circumference is called a *quadrant*.

The straight line joining the extremities of an arc is called a *chord*. The chord is said to *subtend* the arc.



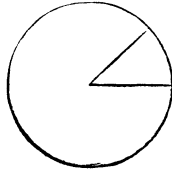
Every chord corresponds always to two arcs, which together make up the entire circumference.

It is the smaller arc which is referred to as the subtended arc, unless otherwise expressed.

The portion of a circle included by an arc and its chord is called a *segment*.

The portion included between two radii and the intercepted arc is called a *sector*.

IV. When a straight line cuts the circumference of a circle it is called a *secant*.



When a straight line touches the circumference in only one point it is called a *tangent*; and the common point of the line and circumference is called the *point of contact*.

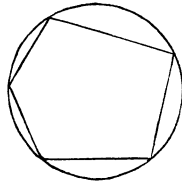
Two circumferences are tangent to each other when they have only one point in common.

Two circumferences are *concentric* when they have the same centre.

V. A line is *inscribed* in a circle when its extremities are in the circumference.

An angle is inscribed in a circle when its sides are inscribed.

A polygon is inscribed in a circle when its sides are inscribed; and under the same circumstances, the circle is said to *circumscribe* the polygon.



A circle is inscribed in a polygon when its circumference touches each side, and the polygon is said to be *circumscribed about the circle*.

By an *angle in a segment* of a circle, is to be understood an angle whose vertex is in the arc, and whose sides intercept the chord of said arc; and by an *angle at the centre*, is meant one whose vertex is at the centre. In both cases the angles are said to be *subtended* by the chords or arcs which their sides include.

VI. Any polygonal figure is said to be *equilateral* when all its sides are equal; and it is *equiangular* when all its angles are equal.

Two polygons are said to be *mutually equilateral* when their corresponding sides, taken in the same order, are equal. When this is the case with the corresponding angles, the polygons are said to be *mutually equiangular*.

A *regular polygon* has all its sides equal, and all its angles equal. If all the sides are not equal, or all the angles are not equal, the polygon is *irregular*.

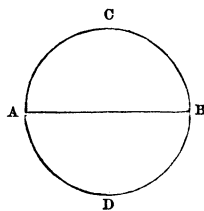
A regular polygon may have any number of sides not less than three. The *equilateral triangle* is a regular polygon of three sides. The *square* is also a regular polygon of four sides.

OF CHORDS, SECANTS, AND TANGENTS.

THEOREM I.

Every diameter divides the circle and its circumference into two equal parts.

Revolve the portion ACB about the diameter AB as a hinge, until it returns to its primitive plane, on the opposite side of AB; then will the portion of the circumference ACB wholly coincide with ADB. For, if not, there would be points in the circumference unequally distant from the centre, which is impossible (D. II.). Hence, the diameter divides the circle and its circumference into two equal parts.



THEOREM II.

A straight line cannot meet the circumference in more than two points.

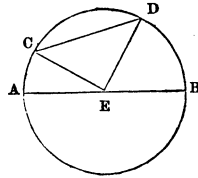
For if it could, drawing radii to these points, we should have more than two equal lines drawn from the same point to a straight line, which is impossible (B. I., T. XII., C. II.).*

* When a reference is made from one Book to another Book, the Book referred to will be given as above; but when the Book is not given, the reference is confined to the Book in which the proposition occurs.

THEOREM III.

The diameter of a circle is greater than any chord.

The diameter AB is greater than any chord, as CD. For, drawing the radii EC and ED, we have $EC + ED > CD$ (B. I., T. VII.). But the diameter $AB = EC + ED$ (D. II.); hence, $AB > CD$.



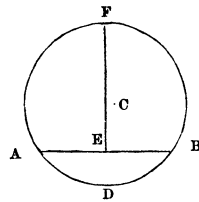
THEOREM IV.

The radius drawn perpendicular to a chord, bisects the chord, and also bisects its subtended arc.

Let the radius CD be drawn perpendicular to the chord AB.

Produce DC to F, and apply the figure DAF to DBF by revolving it about DF. Then the arc DAF will coincide with DBF (T. I.). And since the angles DEA and DEB are right, the line EA will coincide with EB.

Hence we have $AE = EB$, arc $AD = \text{arc } DB$. Also we have arc $AF = \text{arc } FB$.



Scholium. The straight line CD fulfils four different conditions: 1st, it passes through the centre; 2d, it passes through the middle of the chord; 3d, it passes through the middle of the subtended arc; 4th, it is perpendicular to the chord. Any two of these conditions are sufficient to determine the direction of the line. Hence we also have the following propositions:

A radius drawn bisecting a chord is perpendicular to it, and bisects the subtended arc.

A radius drawn bisecting an arc, will also bisect its chord perpendicularly.

A line drawn bisecting a chord and its subtended arc, will pass through the centre, and be perpendicular to the chord.

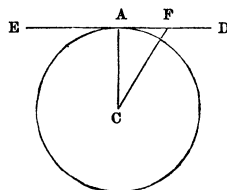
A perpendicular bisecting a chord, will also bisect its subtended arc, and pass through the centre.

A line drawn from the middle of an arc perpendicular to its chord, will bisect the chord, and pass through the centre.

THEOREM V.

A line perpendicular to a radius at its extremity, is tangent to the circumference.

If ED is drawn perpendicular to the radius CA, at its extremity, it will be tangent to the circumference; that is, it will have only the point A in common with the circumference. For any other point, as F, being joined with the centre, gives an oblique line greater than the radius (B. I., T. XII.). Hence, A is the only point common to the straight line and the circumference; consequently, this line is tangent to the circumference (D. IV.).



Cor. I. If a straight line is tangent to the circumference of a circle, it will be perpendicular to the radius drawn to the point of contact.

For all other points of the tangent line, except that of contact, are situated without the circumference, and therefore at a greater distance from the centre than the radius. Hence the radius, being the shortest line which can be drawn from the centre to the tangent, is perpendicular to it (B. I., T. XII.).

Cor. II. Only one tangent can be drawn through the same point of the circumference.

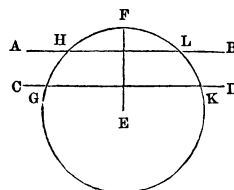
Cor. III. From a point within the circumference no tangent can be drawn.

Cor. IV. The perpendiculars drawn at the extremities of a diameter will be parallel tangents, and conversely two parallel tangents will have their points of contact situated at the extremities of the same diameter.

THEOREM VI.

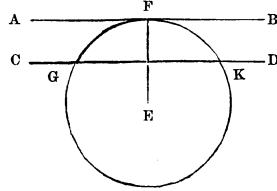
Parallel secants or tangents intercept equal arcs of the circumference.

First. When the parallels are both secants. Draw the radius EF perpendicular to AB, and it will also be perpendicular to its parallel CD, and we shall have (T. IV.) $\text{arc GF} = \text{arc KF}$ and $\text{arc HF} = \text{arc LF}$. Consequently, $\text{arc GF} - \text{arc HF}$

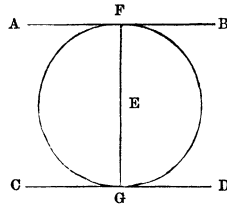


$= \text{arc } KF - \text{arc } LF$; that is, the intercepted arcs GH and KL are equal.

Secondly. When one of the parallels is a secant and the other a tangent. Draw the radius EF to the point of contact, and it will be perpendicular to the tangent (T. V., C. I.), and consequently perpendicular to its parallel CD . Hence the $\text{arc } GF = \text{arc } KF$ (T. IV.).



Thirdly. When both parallels are tangents. The line FG , joining the points of contact, will be a diameter (T. V., C. IV.), consequently the intercepted arcs will be semi-circumferences, and therefore equal.



Cor. Conversely, if two secants intercept equal arcs of the same circumference they will be parallel, provided they do not intersect, within the circumference.

If a secant and a tangent intercept equal arcs of the same circumference, they will be parallel.

If two tangents intercept equal arcs of the same circumference they will be parallel.

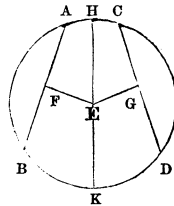
THEOREM VII.

In the same circle, or in equal circles, if two arcs are equal, they will have equal chords, which will be equally distant from the centre.

If the arcs are unequal and each less than the semi-circumference, the greater arc will have the greater chord, which will be nearer the centre.

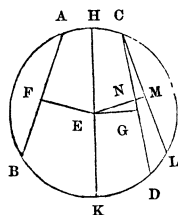
Two circles having equal radii, by superposition, may be made exactly to coincide. Hence we may confine our demonstration to the case of chords in the same circle.

First. Suppose the $\text{arc } AB = \text{arc } CD$. From the centre E draw EF and EG perpendicular to the chords AB and CD . Also through H , the middle point of the arc AC , draw the diameter HK . If we



revolve the portion of the figure on the left of HK over upon the portion on the right, the arcs HA and HAB will coincide with their equal arcs HC and HCD, consequently the chords AB and CD will be equal, and they will have the same perpendicular; that is, EF will equal EG, and the chords will be equally distant from the centre.

Secondly. Suppose the arc AB less than CD. Applying the portion of the figure on the left of HK upon the portion on the right, as before, we see that the point B will fall at some point L, between C and D, since the arc AB is less than CD. Hence, the middle of the arc CL, which is in the prolongation of the perpendicular EM, will be nearer C than the middle of the arc CD which is in the prolongation of the perpendicular EG; consequently the point G is between D and N, the point where the perpendicular EM intersects CD. Therefore CG, the half of CD, is greater than CN, but CN being an oblique line in reference to EM is greater than the perpendicular CM; consequently, $CD > CL$, or $CD > AB$. That is, the greater arc has the greater chord.



Again, we have $EG < EN$, and $EN < EM$, consequently $EG < EM$, or $EG < EF$; that is, the chord corresponding to the greater arc is nearer the centre.

Cor. Conversely, in the same circle, or in equal circles :

1. If the chords are equal, the arcs will be equal. And these chords will be equally distant from the centre.

2. The greater chord will correspond to the greater arc. And the greater chord will be nearer the centre.

We may also add :

3. Chords equally distant from the centre are equal, and subtend equal arcs.

4. Of two chords unequally distant from the centre, the one nearer the centre is the greater, and it subtends the greater arc.

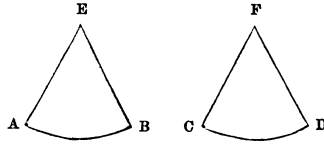
Scholium. We must keep in mind that the arcs considered in this Theorem are in each case less than a semi-circumference.

OF THE MEASURE OF ANGLES.

THEOREM VIII.

In the same circle, or equal circles, equal angles at the centre correspond to equal arcs. Conversely, equal arcs correspond to equal angles.

First. Suppose the angle at E to equal the angle at F. Apply the angle AEB upon the angle CFD; EA and EB will take, respectively, the directions of FC and FD; and since the radii are equal, the points A and B will coincide respectively with the points C and D. Hence the arc AB = arc CD.



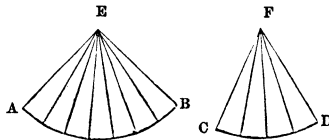
Secondly. Suppose the arcs AB and CD equal. Then, as before, applying the figure ABE upon CDF, so that the radius AE may coincide with its equal radius CF, then will the arcs AB and CD coincide, since they are equal, and the point B will coincide with the point D; consequently the angle AEB is equal to the angle CFD.

Cor. In the same circle, or equal circles, the greater angle at the centre corresponds to the greater arc, and, conversely, the greater arc corresponds to the greater angle.

THEOREM IX.

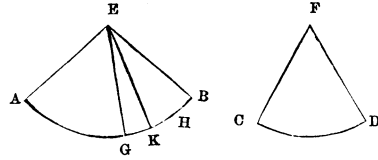
In the same circle, or in equal circles, angles at the centre are to each other as their included arcs.

First. When the two arcs AB and CD are commensurable—that is, when they are to each other as two whole numbers, for instance, as 7 to 4. If we divide the arc AB into seven equal portions, and the arc CD into four, and draw radii to these points of division, we shall divide the angle AEB into seven angles and CFD into four, and these angles will be equal, since their arcs are (T. VIII.). Hence the



angle AEB will be to the angle CFD as 7 to 4; that is, as the arc AB to the arc CD.

Secondly. When the arcs AB and CD are not commensurable. Apply the smaller angle CFD upon the larger angle AEB, so that it may take the position AEG. If now, these angles are not to each other as their arcs, suppose we have



$$\text{angle AEB} : \text{angle AEG} :: \text{arc AB} :$$

arc AH greater than arc AG.

If we conceive the arc AB to be divided into equal portions, each of which shall be less than arc GH, there will be at least one point of division between G and H, as at K. Draw the radius EK, and then since the arcs AB and AK are commensurable, we have by the first part of this Theorem,

$$\text{angle AEB} : \text{angle AEK} :: \text{arc AB} : \text{arc AK}.$$

From these two proportions, since their antecedents are the same, we deduce

$$\text{angle AEG} : \text{angle AEK} :: \text{arc AH} : \text{arc AK},$$

which cannot be, for the antecedent of the first couplet is less than its consequent, while in the second couplet the antecedent is greater than its consequent.

Hence the angle AEB cannot be to the angle AEG as the arc AB is to an arc greater than AG. And by a similar process we may show that the angle AEB cannot be to the angle AEG as the arc AB is to an arc less than AG. Consequently we must have in the same circle, or in equal circles, the angles at the centre to each other as their corresponding arcs.

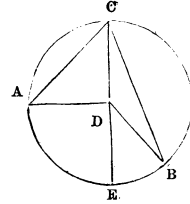
Scholium. Since, in the same circle, or in equal circles, the angles at the centre are to each other as their corresponding arcs, we may use the arcs as the measure of their corresponding angles. In the case of a right angle, its measuring arc is a quadrant; in the case of an angle equal to two right angles, the measuring arc is a semi-circumference. We sometimes regard the right angle as the unit angle, so that acute angles would all be less than 1, and obtuse angles would exceed 1 and be less than 2.

THEOREM X.

An inscribed angle is measured by half the arc included between its sides.

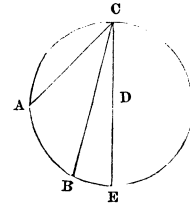
First. When the centre of the circle is within the angle.

Draw the diameter CE and the radii DA and DB . Now, since the triangle DAC is isosceles, the exterior angle ADE is double the angle DCA . The angle ADE is measured by the arc AE (T. IX., S.); hence the angle ACE is measured by one half of the arc AE . In the same way we may show that the angle ECB is measured by one half of the arc EB ; consequently the angle ACB is measured by one half its included arc AEB .

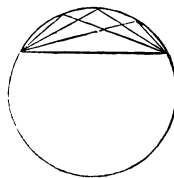
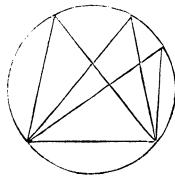
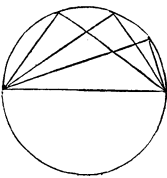


Secondly. When the centre of the circle is without the angle.

As before, draw the diameter CE , and we shall have the angle ACE , measured by one half of the arc ABE , also the angle BCE measured by half the arc BE ; consequently, their difference, the angle ACB , is measured by half the difference of these arcs; that is, by half the included arc AB .



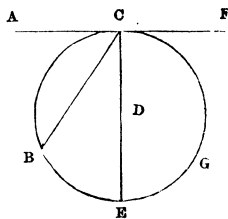
Cor. I. Each angle inscribed in a semicircle is a right angle, since it is measured by half the semi-circumference, or by a quadrant.



Each angle in a segment greater than a semicircle is acute, since half the included arc, which measures it, is less than a quadrant.

Each angle in a segment less than a semicircle is obtuse, since half the included arc, which measures it, is greater than a quadrant.

Cor. II. An angle, ACB , formed by a tangent and chord, has also for its measure one half the arc CB included between its sides. For, drawing the diameter CE , we have ACE a right angle (T. V., C. I.), and it has for its measure one half of the semi-circumference CBE . The angle BCE has for its measure one half the arc BE ; hence, the difference of these angles; that is, the angle ACB has for its measure one half the arc CB .

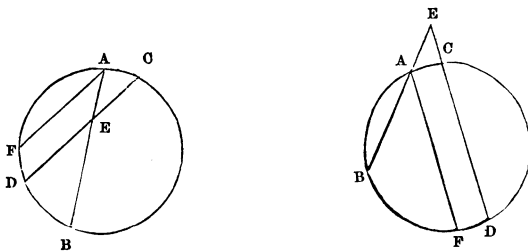


If we consider the supplementary angle FCB , we have the right angle FCE measured by one half the arc CGE ; hence, the sum of the angles FCE and ECB ; that is, the angle FCB has for its measure one half the included arc $CGEB$.

THEOREM XI.

The angle formed by the intersection of two chords is measured by half the sum of the included arcs. And the angle formed by the intersection of two secants is measured by half the difference of the included arcs.

Drawing the chord AF parallel to CD , we have the arc $FD = AC$ (T. VI.), and the angle $BAF = BED$ (B. I., T. XVIII., C.), but the angle BAF is measured, in the case of the chords, by

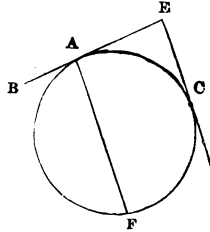


half the sum of the arcs BD and DF ; that is, by half the sum of BD and AC . But in the case of the secants, the angle BAF is measured by half the difference of BD and DF ; that is, by half the difference of BD and AC . Consequently, the angle formed by the intersection of two chords is measured by half

the sum of the included arcs; and the angle formed by the intersection of the two secants is measured by half the difference of the included arcs.

Cor. The angle AEC formed by the intersection of two tangents, is measured by half the difference of the concave and convex arcs AFC and AC, comprehended between the points of contact.

For, drawing AF parallel to CE, we have the arc $FC = AC$ (T. VI.), and the angle $BAF = BEC$, but the angle BAF is measured by half the arc AF (T. X., C. II.); that is, by half the difference of AFC and FC, or which is the same, by half the difference of the concave arc AFC and the convex arc AC.

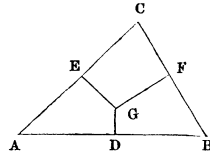


OF INSCRIBED AND CIRCUMSCRIBED POLYGONS.

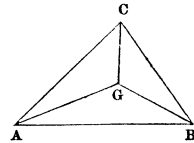
THEOREM XII.

All triangles are capable of being inscribed in a circle and of circumscribing a circle.

First. The three perpendiculars bisecting the three sides of the triangle ABC, meet in the same point G, which is equally distant from A, B, and C (B. I., T. XXXVII.). Hence, if with G as a centre, a circumference be described with a radius equal to the distance from G to either angle of the triangle, it will circumscribe the triangle, and consequently the triangle will be inscribed in the circle.



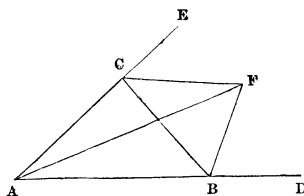
Secondly. The three lines bisecting the three angles of the triangle ABC meet in the same point G, which is equally distant from the three sides of the triangle (B. I., T. XXXVI.). Hence, if with G as a centre, a circumference be described with a radius equal to the distance from G to either side of the triangle, it will be inscribed in the triangle, and consequently the triangle will circumscribe the circle.



Scholium I. Since, in the first case, the perpendiculars can meet in only one point, there can be only one circumference circumscribing a triangle. That is, through three points not in the same straight line, only one circumference can be made to pass. Hence, when two circumferences have three points common, they must coincide.

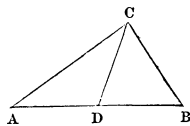
If the three points A, B, and C, are all in the same straight line, the perpendiculars drawn bisecting AB, BC and CA will be parallel, and cannot meet, in which case there can be no centre, unless we regard the centre as at an *infinite* distance.

Scholium II. If, in the second case, the sides AB and AC are produced, the point F, where the lines bisecting the exterior angles meet, is also equally distant from the three lines forming the triangle. Therefore, with F as a centre, a new circumference may be described tangent to the three sides. This



circle is called the *escribed* circle. Hence, if lines be drawn bisecting the angles, and the exterior angles of a triangle, they will intersect each other by threes at the centres of the *inscribed* and *escribed* circles; thus the three lines which bisect the angles will meet at the same point, giving the centre of the inscribed circle. Any one of the lines bisecting an angle of the triangle will intersect, at the same point, two of the lines which bisect the exterior angles, giving the centre of an escribed circle.

Scholium III. This Theorem, taken in connection with Theorem XXXV., Book First, shows that, when the triangle is right-angled at C, the centre of the circumscribed circle will be at D, the middle point of the hypotenuse, for in this case the three lines DA, DB, and DC are all equal.



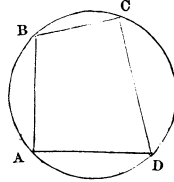
If the triangle is isosceles, the centre of the circumscribing circle as well as that of the inscribed circle will be in the line bisecting the angle formed by the equal sides.

If the triangle is equilateral, the centres of the two circles will coincide, and the circumscribed and inscribed circles will then be concentric.

THEOREM XIII.

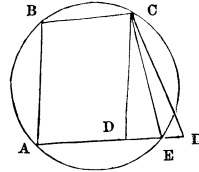
In any inscribed quadrilateral, the sum of the opposite angles is equal to two right angles.

The angle at B is measured by half the arc CDA, and the opposite angle at D is measured by half the arc ABC; hence the sum of the angles at B and D is measured by half the entire circumference, which is the measure of two right angles.



Cor. Conversely, a quadrilateral may be inscribed when the sum of its opposite angles is equal to two right angles.

If we describe a circumference through the three points A, B, and C, which can always be done (T. XII., S. I.), and it does not pass through D, then the point D must be either within or without this circumference. First suppose it within. Produce AD until it meet the circumference at E, then; by the Theorem itself, we shall have the sum of ABC and CEA = 2 right angles: but by hypothesis $ABC + CDA = 2$ right angles; consequently $CEA = CDA$, which is absurd (B. I., T. VI., C.). Hence, the point D cannot be within the circumference. By a similar process, we can show that it cannot be without the circumference. It must therefore be in the circumference.

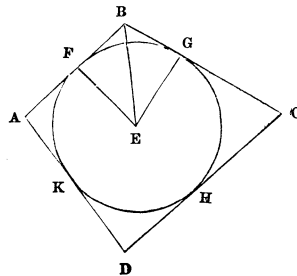


Scholium. The *rectangle*, which includes the *square*, is the only parallelogram which can be inscribed in a circle. The diagonals of the rectangle are diameters of the circumscribing circle.

THEOREM XIV.

When a quadrilateral circumscribes a circle, the sums of the opposite sides are equal.

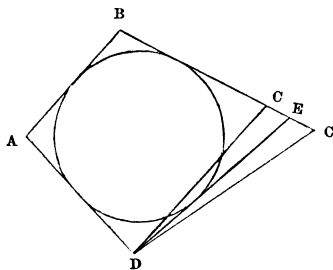
From the centre draw EF and EG to the points of contact of the sides AB and BC, and they will be perpendicular to those sides (T. V., C. I.). The two right-angled triangles EFB,



EGB, will be equal, since they have the hypotenuse and a side of the one respectively equal to the hypotenuse and a side of the other (B. I., T. XXVI.), and $BF = BG$. In a similar manner we may show that $AF = AK$; $CH = CG$; $DH = DK$. Consequently by adding, we have $AB + CD = BC + AD$.

Cor. When the sums of the opposite sides of a quadrilateral are equal, it is capable of circumscribing a circle.

For, if we describe a circumference tangent to AD , AB , and BC , which can always be done (T. XII., S. I. or II.), it will also be tangent to DC . For, if the side DC is not tangent to the circumference, it must be either a secant or lie wholly without the circumference. We will first suppose it to be a secant. If DE



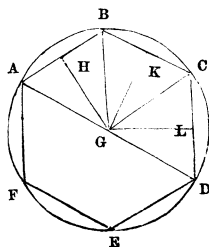
is drawn tangent to this circumference, we shall have $AD + BE = AB + DE$, but by hypothesis we have $AD + BC = AB + DC$; hence, by subtraction, we obtain $EC = DE - DC$; that is, one side of a triangle is equal to the difference of the other two sides, which is impossible (B. I., T. VII.). Hence the side DC cannot be secant. In a similar way we can show that it cannot be wholly without the circumference. It must therefore be tangent.

Scholium. The *rhombus* and the *square* are the only quadrilaterals capable of circumscribing a circle.

THEOREM XV.

All regular polygons are capable of being inscribed in a circle, and of circumscribing a circle.

Let $ABCDEF$ be any regular polygon; and through three consecutive angles A , B , and C , describe the circumference of a circle (T. XII.). This circumference will also pass through all the other angles of the polygon. For, drawing from the centre G , the lines GA , GB , GC , and GD , we shall form three triangles, GAB , GBC ,



GCD. In the two triangles GAB and GBC, we have GB common, $AB = BC$ and $GA = GC$; hence the triangles are equal (B. I., T. XXV.), and being isosceles since $GA = GB = GC$, we have the angles $GAB = GBA = GBC = GCB$. That is, each of these angles is one half the angle of the polygon; hence the angle $GCB = GCD$. Now, comparing the two triangles GBC and GCD, we have the side GC common, $BC = CD$, and the angle $GCB = GCD$; these triangles are therefore equal (B. I., T. XX.); and we have $GD = GB = GC = GA$; hence, the circumference which passes through A, B, and C, will also pass through D.

In a similar manner we can show that the circumference which passes through B, C, and D, will also pass through E, and so on, for all the angles of the polygon, which demonstrates the first part of the Theorem.

As to the second part, we observe, that the sides AB, BC, CD, etc., are equal chords, in reference to the circumscribed circle, and therefore they are equally distant from the centre G (T. VII., C. I.). If then, with G as a centre, and with the perpendicular GH as a radius, we describe a circumference, it will be tangent to all the sides of the polygon. This circle will be inscribed in the polygon, and the polygon will circumscribe the circle.

Scholium. The point F, the common centre of the inscribed and circumscribed circles, is called the *centre of the regular polygon*.

The radius of the circumscribing circle is called the *radius of the polygon*, and the radius of the inscribed circle is called the *apothem*.

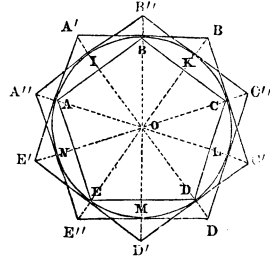
The angles AGB, BGC, CGD, etc., are called *angles at the centre*.

The lines GA, GB, GC, etc., which are the radii of the polygon, bisect the angles of the polygon. The apothems GH, GK, GL, etc., which are the radii of the inscribed circle, bisect the sides of the polygon perpendicularly, and also bisect the angles at the centre, since they would, if produced, bisect the arcs which measure the angles at the centre.

THEOREM XVI.

If through the corners of a regular polygon, already inscribed in a circle, we draw tangents, we shall thus circumscribe a regular polygon of the same number of sides.

The tangents thus drawn will form with the sides of the inscribed polygon, AB , BC , CD , etc., a series of triangles, $AA'B$, $BB'C$, $CC'D$, etc., all isosceles and equal, since $AB = BC = CD = \text{etc.}$, and the angles $A'AB$, $A'BA$, $B'BC$, $B'CB$, $C'CD$, $C'DC$, etc., are all equal, having for their measure half of the equal arcs AB , BC , CD , etc. (T. X., C. II.). Hence, all the angles, A' , B' , C' , etc., of the circumscribed polygon are equal. And since $AA' = A'B = BB' = B'C = CC' = \text{etc.}$, we have $A'B' = B'C' = C'D' = \text{etc.}$ That is, this circumscribed polygon has all its angles equal, and all its sides equal; it is therefore regular (D. VI.).



Scholium. Since the radius OA' of the circumscribed polygon bisects the angle $E'A'B'$ (T. XV., S.), we have, in the two right-angled triangles OAA' and OBA' , the acute angle $OA'A$ of the one, equal to the acute angle $OA'B$ of the other; consequently, the remaining acute angles are equal; that is, the angle $A'OA = A'OB$. In the same way we can show that the angle $B'OB = B'OC$. But the angle $A'OB = B'OB$, since the apothem FB bisects the angle $A'FB'$ at the centre (T. XV., S.). Hence, all the angles AOA' , $A'OB$, BOB' , $B'OC$, COC' , etc., are equal; and their measuring arcs AI , IB , BK , KC , CL , etc., are all equal.

If then we suppose the circumscribed polygon to revolve in its own plane about its centre O , as a pivot, so that A may reach the point I , B will then coincide with K , C with L , etc. The circumscribed polygon will by this means, without having its relative parts in the least changed, assume a new position, having its sides parallel with the sides of the inscribed polygon.

OF SECANT AND TANGENT CIRCLES.

THEOREM XVII.

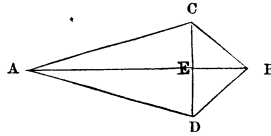
If two circumferences of circles have two points in common, the line joining their centres will bisect their common chord perpendicularly.

For, the line bisecting a chord at right angles passes through the centre (T. IV., S.); and as this chord is common to both circles, this bisecting line must pass through both centres.

THEOREM XVIII.

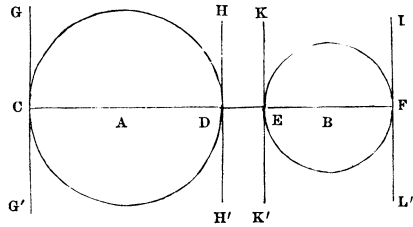
When two circumferences have only one point in common, this point will be situated on the line joining their centres.

If A and B are the centres of two circles having only one point in common, that point will be situated on the line AB joining their centres. For, if not, suppose it to be situated without this line, as at C. Draw CE perpendicular to AB, and prolong it until $ED = CE$; then we evidently have $AC = AD$, also $BC = BD$, and since C was a common point of the circumferences, D will also be a common point. These circumferences will then have two points in common, which is contrary to the hypothesis. Hence, the point in common to the two circumferences must be in the line joining their centres.

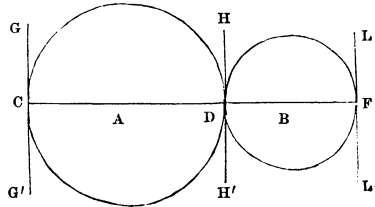


Scholium. Suppose we consider two circles whose centres are at A and B, the one at B being the smaller. Draw CF passing through their centres, and at the extremities of their diameters, CD and EF, draw the perpendiculars GG', HH', KK', LL'. If we suppose the circle, whose centre is at A, fixed, and the other to move towards it, so that the centre B may move in the line AB, we may notice *five distinct positions* of these circles.

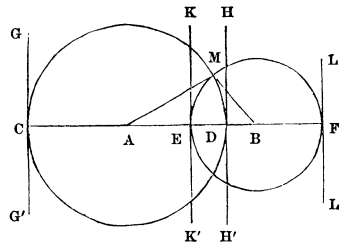
First. When the tangent KK' is on the right of HH' . In this case, each circle will be wholly without the other, and they have no common point.



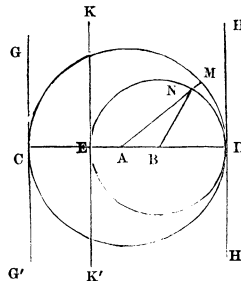
Secondly. When the tangent KK' coincides with HH' , and becomes a common tangent to the two circumferences. In this case, the point D is the only point common to the two circumferences, and the circles are said to touch *externally*.



Thirdly. When the limiting tangents of the smaller circle are on opposite sides of HH' . In this case, there will be a portion of space common to the two circles; hence their circumferences must necessarily cut each other in two points.



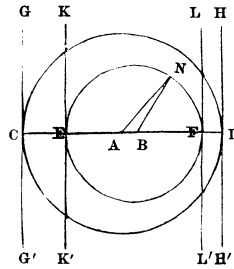
Fourthly. When the tangent LL' coincides with HH' , and becomes a common tangent to the two circumferences. In this case, D is the only point common to the two circumferences. For, drawing any radius, AM , of the larger circle, cutting the circumference of the smaller circle at N , and we shall have



$$AN < AB + BN, \text{ or } AN < AB + BD, \text{ or } AN < AD, \text{ or } AM.$$

Hence, every point in the circumference of the smaller circle, except the point D , is within the circumference of the larger circle. The circles are said to touch *internally*.

Fifthly. When the limiting tangents of the smaller circle are both situated between those of the larger circle. In this case the two circumferences have no common point, and the smaller is wholly within the larger. For, joining any point of its circumference, as N , with the centres, we have



$$AN < AB + BN, \text{ or } AN < AB + BF,$$

but $AB + BF = AF$, which is less than AD ; hence, $AN < AD$.

THEOREM XIX.

When two circumferences have no point in common, the distance between their centres is either greater than the sum of their radii, or less than their difference.

For by the Scholium under the last Theorem we have two positions of the circumferences in which they have no common point.

First. When they are, the one wholly without the other, in which case the distance between their centres is greater than the sum of their radii.

Secondly. When the smaller circumference lies wholly within the larger; in which case the distance between their centres is less than the difference of their radii.

If we denote the greater radius by R , the less radius by R' , and the distance between their centres by D , we shall have, when the circles are *exterior*,

$$D > R + R',$$

and when they are *interior*—that is, when the one is wholly within the other—we shall have

$$D < R - R'.$$

THEOREM XX.

When two circumferences are tangent to each other, the distance between their centres is equal to the sum of their radii, or equal to their difference.

There are two positions of the circumference in which

they are tangent to each other (T. XVIII, S.). In the first case,

$$D = R + R',$$

and in the second case,

$$D = R - R'.$$

THEOREM XXI.

When two circumferences cut each other, the distance between their centres is less than the sum of their radii, and greater than their difference.

By the third case under the Scholium of Theorem XVIII. (see figure of the same), we see that the point M, common to the two circumferences, must be necessarily without the line AB which joins their centres; hence the three points A, B, and M must form a triangle, consequently we must have

$$D < R + R' \text{ and } D > R - R'.$$

Scholium. The reciprocals of the foregoing Theorems may readily be demonstrated by the method of reducing to an absurdity. They may be thus stated:

When two circumferences are in the same plane:

1. If $D > R + R'$, the two circles are exterior the one to the other.
2. If $D = R + R'$, the two circles will touch externally.
3. If $D < R + R'$, and at the same time $D > R - R'$, the two circles will cut each other in two points.
4. If $D = R - R'$, the two circles will touch internally.
5. If $D < R - R'$, the smaller circle will lie wholly within the other.

PROBLEMS,

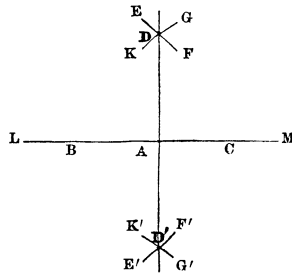
WHICH REFER TO THE FIRST AND SECOND BOOKS.

OF PERPENDICULARS, ANGLES, AND PARALLELS.

PROBLEM I

From any given point of an indefinite straight line, to draw a perpendicular to this line.

Let A be the given point of the indefinite line LM. With the compass, take two equal distances, AB and AC. With A and B as centres, and with a radius greater than AB or AC, describe two circumferences intersecting each other at D. Draw DA, and it will be the perpendicular required.



For, by construction, the two circumferences described are such that the distance of their centres BC is less than the sum of their radii, since each radius is greater than half this distance, and it is at the same time greater than the difference of their radii, which is zero; hence these circumferences cut each other in some point, as D (T. XXI., S.); and if we draw DA, it must be perpendicular to BC (B. I., T. XIII., C. II.).

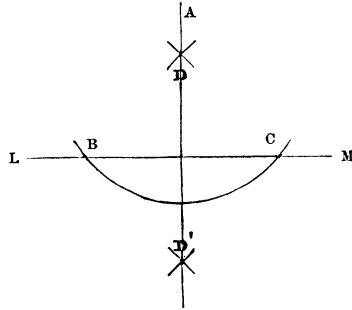
Scholium. The two circumferences cut each other in a second point D', on the other side of the line LM, which point may be used to verify our work, since the three points D, A, D' must be in the same straight line.

In practice, instead of describing the entire circumferences, we trace only the arcs EF, GK, E'F', G'K', in the immediate vicinity of the points of intersection.

PROBLEM II.

From a given point without a straight line, to draw a perpendicular to this line.

Let A be the given point without the line LM. With A as a centre and with a radius sufficiently great, describe an arc cutting LM in two points B and C; and with B and C as centres, and with a new radius greater than half of BC, describe two new arcs cutting each other in D and D'; draw ADD', and it will be the perpendicular required. For the points A, D, and D' are, by construction, equally distant from B and C, consequently the line ADD' is perpendicular to BC or LM (B. I., T. XIII., C. II.).

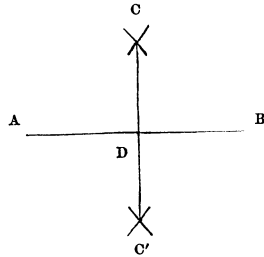


PROBLEM III.

To divide a given straight line of a determinate length into two equal parts; that is, to bisect a given line.

Let AB be the line which we wish to bisect.

With A and B as centres, and with a radius greater than half of AB, describe arcs cutting each other in C and C', draw CC', and it will bisect AB perpendicularly at D. The same demonstration as in preceding problems applies.



Cor. The following problems follow immediately from the foregoing:

First. To describe on a given straight line as a diameter a semi-circumference, or the entire circumference. For this question is reduced to finding the middle point of this line.

Secondly. To describe a circumference through three given

points. For, having joined these points, two and two, the whole is reduced to drawing perpendiculars through the middle points of these lines. Where these perpendiculars meet, will be the centre of the circle. The radius will be found by joining the centre and either of the given points.

Thirdly. To find the centre of a circle, or of an arc already described. For, taking at pleasure any three points on the arc and proceeding as in the last case, we shall find the centre.

Fourthly. To divide an arc into two equal parts. For, draw the chord, and then draw a perpendicular through its middle point (T. IV., S.).

PROBLEM IV.

To draw from a given point a perpendicular to a given line which can be produced only in one direction.

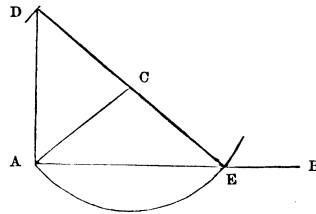
First. When the given point A is in the given line AB, which line we will suppose cannot be prolonged to the left of A.

Take any point, as C, for a centre, and with CA as a radius describe an arc cutting AB at E.

Draw EC, and produce it until CD is equal to CE. Then draw AD, it will be perpendicular to AB. For, by construction, $CA = CE = CD$, and the angle DAB is right (B. I., T. XXXV.).

Secondly. When the given point D is without the given line AB.

Draw any line, as DE, meeting AB and with C its middle point as a centre, and with $CE = CD$ as a radius, describe an arc cutting AB in A. Join A and D, and it will be the perpendicular required. For, by construction we have, as before, $CA = CE = CD$; consequently, DAE is a right angle

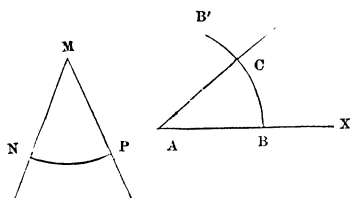


PROBLEM V.

Through a given point in an indefinite straight line to draw a second line, forming with the first a given angle.

Let M be the given angle, and A the given point in the in-

definite line AX. With M and A as centres with a radius $MN = AB$, describe the arc NP comprised between the sides of the given angle and the indefinite arc BB' . Then with B as

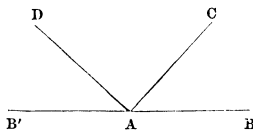


a centre, and with a radius equal to the chord of the arc NP, describe a new arc cutting the indefinite arc BB' in C. Draw AC, and BAC will be the angle required. For, by construction, the chords of the two arcs BC and NP are equal; hence the two arcs are equal, and consequently the angles are equal (T. VIII.).

Scholium I. To divide an angle into two, four, eight, etc., equal parts, it is sufficient to divide the corresponding arc first into two equal parts, and afterwards to divide each half into two equal parts, and so on.

Scholium II. The same problem furnishes the means of determining the supplement of the sum of two given angles, or, in other words: Knowing two angles of a triangle, to determine the third.

After forming the angle BAC equal to the first of the given angles, draw AD, making the angle CAD equal to the second of the given angles. Then produce BA to B' and DAB' will be the supplement required.

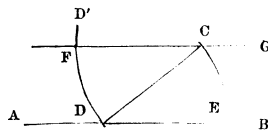


PROBLEM VI.

From a point without a straight line to draw a parallel to this line.

Let C be the given point, and AB the given line.

Draw any line from C meeting AB, as CD; and with C and D as centres with CD for a radius, describe the indefinite arc DD' , and the definite arc CE terminating at E in the given line AB. Take the arc DF equal to CE, and draw CF, which will be the parallel required.

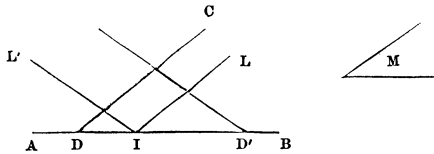


For, by construction, we have angle $FCD = \text{angle } CDE$ (T. VIII.), hence FG and AB are parallel (B. I., T. XVII., C.).

Cor. Problems V. and VI. afford the means of solving the following:

Through a given point, C, without a straight line AB, to draw a second line meeting the first under a given angle M.

For, at any point, I, of the line AB, draw IL, making the angle LIB = angle M; then through C draw CD parallel to LI, which is evidently the line required. Since from any point, I, two lines IL and IL' may be drawn, making with AC an angle equal to M, if M is not a right angle, it follows that the problem admits of two solutions, unless M is a right angle.

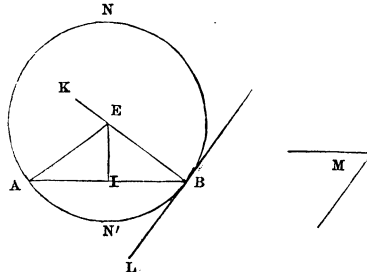


PROBLEM VII.

On a given straight line, as a chord, to describe an arc of a circle capable of containing a given angle.

Let AB be the given line, and M the given angle.

Draw BL, making the angle ABL = M, and draw BK perpendicular to BL. Through I the middle point of AB draw the perpendicular IE meeting BK in E. From the point E with a radius EA = EB, describe a circle. The portion ANB is the arc required. For, by construction, BL is a tangent, being perpendicular to the radius EB, and the angle ABL = M, is measured by half the arc AN'B (T. X., C. II.). But each angle in the segment ANB is also measured by half the same arc (T. X.).



CONSTRUCTION OF POLYGONS.

PROBLEM VIII.

To construct a triangle when we have given :

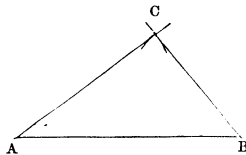
I. *One side and the two adjacent angles.*

II. *Two sides and the included angle.*

III. *The three sides.*

It is not necessary to give at length the construction of the first and second cases, since they are so readily deduced from the Corollary to Problem VI. We remark, however, that in the first, in order that the construction may be possible, the sum of the two given angles must be less than two right angles. And in the second case, the construction is always possible, since, after having drawn two lines, forming with each other the required angle, we can always take on these lines portions equal to the given sides.

In the third case. Draw AB equal to one of the given sides, and with A as a centre with a radius equal the second of the given sides, describe an arc ; also, with B as a centre and with a radius equal to the third side, describe an arc intersecting the former arc in C. Draw CA and CB, and the triangle ABC will evidently satisfy the conditions announced.



That this construction may be possible, it is necessary that the first side may be less than the sum of the other two sides, and greater than their difference (B. I., T. VII.).

Scholium. If, in the first case, instead of having the two adjacent angles given, we had one of the adjacent angles and the opposite angle, we could, by Prob. V., Scholium II., find the third angle, and then the two adjacent angles would be known.

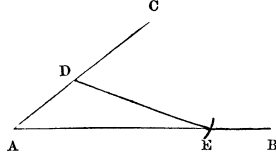
PROBLEM IX.

To construct a triangle when two sides are given and an angle opposite one of these sides.

First. When the side opposite the given angle is greater

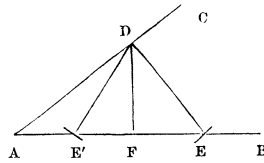
than the adjacent side, the given angle being of any magnitude whatever.

Draw the two lines AB and AC, forming the angle BAC equal to the given angle; take AD equal to the adjacent side, and with D as a centre, and with a radius equal to the opposite side, describe an arc cutting AB in E. Draw DE, and the triangle DAE will fulfil the conditions announced.



Secondly. When the side opposite the given angle is less than the adjacent side, the given angle being acute.

As before, make the angle BAC equal to the given angle, take AD equal to the adjacent side, and with D as a centre, and with a radius equal to the opposite side, describe an arc cutting AB in two points E and E': drawing DE and DE', we have two triangles DAE and DAE' fulfilling the required conditions. If, however, the opposite side is equal to the perpendicular DF, the arc will touch AB, and the two points E and E' will unite with F, and there will be only one triangle, which will be right-angled at F. But if it is less than this perpendicular, the arc cannot either intersect or touch AB, and the construction will be impossible. It is obvious that the construction would be impossible if the given angle were either right or obtuse, so long as the opposite side is less than the adjacent side.



Scholium. It is not necessary to consider the case when the two sides are equal, for the triangle would then be isosceles; and having one angle given, the other angles could readily be found, and then it would be included in Problem VIII.

PROBLEM X.

A regular inscribed polygon being given, to construct a regular circumscribed polygon of the same number of sides; and conversely.

First construction. Through the vertices A, B, C, etc., of the inscribed polygon, draw tangents to the circumference, and they will form the polygon required

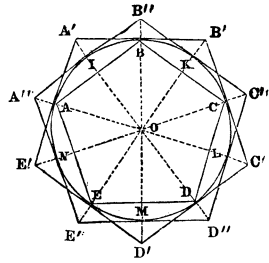
Second construction. Through the extremities I, K, L, etc., of the radii drawn perpendicular to the sides AB, BC, CD, etc., of the inscribed polygon, draw tangents which will form the polygon required.

We have already shown that the two polygons thus obtained are equal (T. XVI., S.).

Reciprocally. To obtain the inscribed polygon, when we already know the circumscribed polygon $A'B'C'D'E'$, we join the consecutive points of contact A, B, C, etc., of the circumscribed polygon. These chords, AB, BC, CD, etc., are equal, since their arcs AIB, BKC, CLD, etc., are equal; and they form with each other equal angles, since the arcs which measure them are equal.

Or we could equally as well use the circumscribed polygon $A''B''C''D''E''$, by joining the consecutive points A, B, C, etc., where the straight lines OA'' , OB'' , OC'' , etc., drawn from the centre to the vertices of the circumscribed polygon, meet the circumference.

The polygon ABCDE is regular, since the angles at the centre $A''OB''$, $B''OC''$, etc., being equal, their arcs which measure them are equal, and consequently the chords of these arcs are equal.

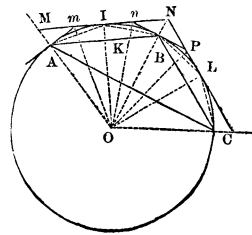


PROBLEM XI.

Having given two regular polygons, the one inscribed and the other circumscribed, each of the same number of sides, to inscribe and circumscribe regular polygons of double the number of sides, and of half the number of sides.

Let AB and MN be sides of the given polygons.

First. To obtain an inscribed polygon of double the number of sides, it is evidently sufficient to join the points A, B, with the point I, the middle of the arc AB, and to repeat the operation for each of the arcs BC, CD, etc.



The chords AI, IB, BL, etc., are equal, since they are subtended by equal arcs.

Secondly. We obtain the corresponding circumscribed polygon, by drawing through the points A, B, two tangents which terminate at the points m, n , on the tangent MN, and repeating the operation at the points C, D, etc.

The line mn is the side of a circumscribed polygon of double the number of sides; and Am, nB are half sides.

In effect, the right-angled triangles OAm, OIm are equal, having the common hypotenuse Om and the side $OA=OI$; consequently $Am=Im$, and the angle $AOm = \text{angle } IOm$.

We see also that the angle mOn is half the angle at the centre of each of the given polygons, and it is therefore the angle at the centre of the polygon sought.

Thirdly. As to the inscribed and circumscribed polygons of half the number of sides, we may obtain them by drawing through the alternate points A, C, E, etc., chords AC, CE, etc. And through the same points drawing tangents, having no regard to the intermediate points B, D, F, etc.

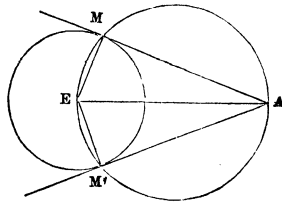
Scholium. It is evidently necessary in the third case, in order that the problem may be possible, to admit that the given polygons have an even number of sides.

OF CONTACT

PROBLEM XII.

Through a given point without the circumference of a circle to draw a tangent to this circumference.

Let E be the centre of the given circle, and A the given point. Draw AE, and upon it, as a diameter, describe a circumference which will cut the given circumference in two points M, M', AM, and AM', being drawn, will be the tangents required. For,



the angles EMA, EM'A being in a semicircle, are right (T., X. C. I.). Consequently, AM and AM' are respectively perpendicular to the radii EM and EM', and therefore tangent to the circumference (T. V.).

Scholium. The case when the given point is in the circumference offers no difficulty, since we then draw a radius to the given point, and through its extremity draw a perpendicular.

OF COMMON MEASURE.

PROBLEM XIII.

To find the common measure of two given lines, provided they have one, and consequently their numerical ratio.

Let AB and CD be the given lines.

From the greater AB cut off parts equal to the less line CD, as many times as possible; for example, twice, with the remainder FB.

From the line CD cut off parts equal to FB, as many times as possible; for example, once, with the remainder GD.

From the first remainder FB cut off parts equal to the remainder GD, as many times as possible; for example, once, with the remainder HB.

From the second remainder GD cut off parts equal to the third remainder HB, as many times as possible; for example, twice, without a remainder.

The last remainder, HB, will be a common measure of the given lines.

If we regard HB as a unit, GD will be 2, and

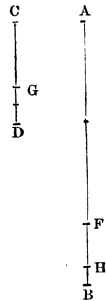
$$FB = FH + HB = GD + HB = 3;$$

$$CD = CG + GD = FB + GD = 3 + 2 = 5;$$

$$AB = AF + FB = 2 CD + FB = 10 + 3 = 13.$$

Therefore the line AB is to the line CD as 13 to 5.

If AB is taken for the unit, CD will be $\frac{5}{13}$; but if CD be taken as the unit, AB will be $\frac{13}{5}$.



If AB is $\frac{1}{2}$ of a yard, then CD will be $\frac{5}{13}$ of $\frac{1}{2}$ a yard, or $\frac{5}{26}$ of a yard.

Again, if CD is $\frac{2}{3}$ of a foot, then AB will be $\frac{13}{5}$ of $\frac{2}{3}$ of a foot = $\frac{26}{15}$ of a foot; and so on for other comparisons.

Cor. Since arcs of the same circle, or of equal circles, are capable of superposition, this same method may be employed when we wish to find a *common measure of two arcs of the same circle, or of equal circles.*

Scholium. Magnitudes frequently have no common measure; that is, they are *incommensurable*; in which case, we shall always, by the foregoing method, have a remainder, however far we carry our successive operations. But these successive remainders would finally become so small that they might practically be neglected. Consequently, we can always find two numbers whose ratio shall approximate as close as we please to the ratio of the given incommensurable magnitudes.

This being admitted, we say that two incommensurable magnitudes A and B are proportional to two other incommensurable magnitudes A' and B', or that the ratio of A to B is equal to the ratio of A' to B', when we obtain for each the same numerical ratio at each successive step of the approximation.

PROBLEM XIV.

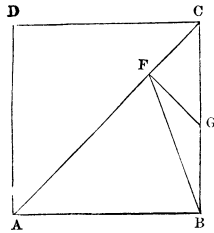
Show, geometrically, that the side of a square and its diagonal are incommensurable.

Let ABCD be the square, and AC its diagonal.

Cutting off AF from the diagonal equal to AB, a side of the square, we have the remainder CF, which must be compared with CB.

If we join FB, and draw FG perpendicular to AC, the triangle BGF will be isosceles.

For, the angle ABG = AFG, each being a right angle; and since the triangle ABF is isosceles, the angle ABF = AFB. Therefore, subtracting the angle ABF from ABG, the remainder FBG will equal the angle BFG, found by subtracting the angle AFB from AFG. Consequently the triangle BGF is isosceles,



and $BG = FG$; but, since AC is the diagonal of a square, the angle FCG is half a right angle; but CFG is a right angle, and consequently FGC is also half a right angle, and CG is the diagonal of a square whose side is CF .

Hence, after $CF = FG = BG$ has been taken once from CB , it remains to take CF from CG ; that is, to compare the side of a square with its diagonal, which is the very question we set out with, and of course we shall find precisely the same difficulty in the next step of the process; so that, continue as far as we please, we shall never arrive at a term in which there will be no remainder. Therefore there is no common measure of the diagonal and side of a square.

THIRD BOOK.

THE PROPORTIONS OF STRAIGHT LINES AND THE AREAS OF RECTILINEAL FIGURES.

DEFINITIONS.

I. *Two figures are similar* when they resemble each other in all their parts, differing only in their comparative magnitudes, so that every point of the one has a corresponding point in the other. If, in two similar figures, we conceive a line joining any two points of the one, and another line joining the corresponding points of the other, the numerical ratio of these lines, which is always the same, is called the *ratio of similitude*.

When the ratio of similitude is equal to unity, the two similar figures become equal in all respects.

II. *Two triangles are similar* when their corresponding sides are proportional.

III. *Two polygons are similar* when they are capable of being decomposed into the same number of similar triangles, each to each, having the same order of arrangement.

IV. In similar figures *homologous points, lines, or angles*, are those which have like positions, in regard to the two figures.

V. The *area* of a surface, or its superficial extent, is the numerical ratio of this surface referred to its *unit surface*. We readily see that figures of very different forms may have the same superficial extent. When we wish to express the property that two surfaces have the same area, without, however, being equal, or capable of superposition, we say they are *equivalent*.

VI. In determining the area of surfaces, it is found convenient to call the *bases* of a parallelogram the sides which are parallel. The *altitude* or *height* is the common perpendicular to these bases, of which one is called the *inferior* or *lower* base, and the other the *superior* or *upper* base.

In the rectangle, two consecutive sides form the *base* and the *height* respectively. In the square, the base and height are equal.

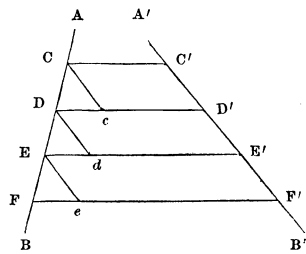
In the case of a triangle, either side may be regarded as the base. The height will be the perpendicular, drawn from the opposite angle to the base, or to the base produced.

PROPORTIONAL LINES.

THEOREM I.

If equal portions are taken on a straight line, and through the points of division parallel lines are drawn meeting a second line, the intercepts of this second line will be equal also.

If the portions CD , DE , and EF , of the straight line AB , are equal; and if through the points C , D , E , and F , the parallels CC' , DD' , EE' , and FF' , are drawn, meeting the line $A'B'$, then will the intercepts $C'D'$, $D'E'$, and $E'F'$ be equal.



For, drawing Cc , Dd , and Ee , parallel to $A'B'$, the triangles CDC , DEd , and $EF e$ will be equal, since the sides CD , DE , and EF are equal, and their corresponding angles are obviously equal (B. I., T. III.). Hence, $Cc = Dd = Ee$, and consequently $C'D' = D'E' = E'F'$ (B. I., T. XXXI., C. II.).

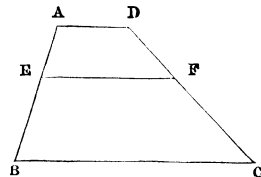
THEOREM II.

If a line is drawn parallel to the bases of a trapezoid, intersecting the sides, it will divide them into proportional parts.

If EF is parallel to AD and BC , we shall have

$$AE : EB :: DF : FC.$$

First. When AE and EB are commensurable; that is, when they are to



each other as two whole numbers, as 7 to 11 for instance, so that

$$AE : EB :: 7 : 11,$$

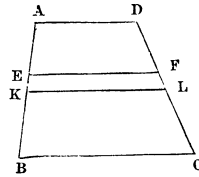
then will DF and FC be to each other in the same ratio.

For, if we conceive the straight line AB divided into $7 + 11 = 18$ equal parts, and through the points of division parallels to BC to be drawn, they will divide DC into 18 equal parts (T. I.), of which 7 will belong to DF, and 11 to FC. Hence,

$$DF : FC :: 7 : 11.$$

Secondly. When AE and EB are incommensurable.

We may conceive AB divided into equal parts so small as to bring K, one of the points of division, as near as we please to E. Then drawing KL, we have, by the first case,



$$AK : KB :: DL : LC.$$

And as this will continue, however small the divisions are, we shall have (B. II., P. XIII., S.)

$$AE : EB :: DF : FC.$$

THEOREM III.

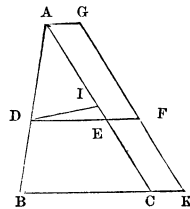
If a line be drawn parallel to either side of a triangle, cutting the other sides, it will divide them into proportional parts.

Although this Theorem might have been made a corollary to the last, we have preferred making it one of our principal Theorems, not only on account of its own importance, but also by reason of the important corollaries which it is capable of furnishing.

Let DE be drawn parallel to the side BC of the triangle ABC, cutting the other sides at D and E; then we shall have

$$AD : DB :: AE : EC.$$

Through A draw AG parallel to BC, and through any point, as G, of this parallel, draw



GK parallel to AC meeting BC produced. Also produce DE to F. We have (T. II.)

$$AD : DB :: GF : FK ;$$

but, by reason of the parallels AC, GK, we have $GF = AE$, $FK = EC$; consequently,

$$AD : DB :: AE : EC.$$

Cor. I. Conversely. If a straight line DE divide two sides of a triangle into proportional parts, it will be parallel to the third side. For, if this is not the case, we might draw through D another straight line DI parallel to BC, and we should then have

$$AD : DB :: AI : IC ;$$

but by hypothesis,

$$AD : DB :: AE : EC ;$$

hence, by equality of ratios, we have

$$AI : IC :: AE : EC,$$

which is absurd, since $AI < AE$, and $IC > EC$.

Cor. II. From the proportion

$$AD : DB :: AE : EC$$

we have, by *composition*,

$$AD + DB : AD :: AE + EC : AE,$$

$$AD + DB : DB :: AE + EC : EC,$$

which gives these new propositions:

$$AB : AD :: AC : AE \text{ and } AB : DB :: AC : EC.$$

Conversely. If a straight line DE is so drawn as to give

$$AB : AD :: AC : AE,$$

we shall have, by *division*,

$$AB - AD : AD :: AC - AE : AE,$$

that is,

$$DB : AD :: EC : AE,$$

which is the same as $AD : DB :: AE : EC$;

consequently, the straight line DE is parallel to BC (C. I.).

Scholium. Since, in every proportion, the product of the extremes is equal to the product of the means, we derive from the above proportion, $AD : DB :: AE : EC$, this equation, $AD \times EC = DB \times AE$.

Now, in order to comprehend the sense which we ought to attribute to the expression, *the product of two lines*, $AD \times EC$, or $DB \times AE$, it is necessary to suppose that the straight lines AD , EC , DB , and AE , have been referred to the same *linear unit*; and instead of lines, properly speaking, we have only to consider, in reality, *abstract numbers* expressing the ratio of these lines to their unit. We ought then, as in Arithmetic, to attach to the word *product* the idea of the multiplication of two numbers.

Hereafter, we may be required to multiply *a surface by a line*, *a surface by a surface*, and even *a volume by a volume*, etc. But in all these cases it must be understood that it is the ratios of these geometrical magnitudes to their respective units, which are thus multiplied.

SIMILAR TRIANGLES.

THEOREM IV.

A straight line drawn parallel to either side of a triangle, intersecting the other two sides, will form a new triangle similar to the first.

If DE is parallel to BC , we shall have

$$AB : AD :: AC : AE :: BC : DE.$$

For, since DE is parallel to BC , we have
(T. III., C. II.)

$$AB : AD :: AC : AE.$$

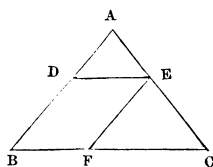
Drawing EF parallel to AB , we also have

$$AC : AE :: BC : BF \text{ or } DE;$$

hence, we have, by equality of ratios,

$$AB : AD :: AC : AE :: BC : DE.$$

Consequently, these triangles are similar (D. II.).



THEOREM V.

Two similar triangles, that is to say, which have their sides proportional, have their homologous angles equal, each to each.

Let ABC and $A'B'C'$ be the two similar triangles.

Take $AB'' = A'B'$, and draw $B''C''$ parallel to BC ; the two triangles $A'B'C'$, $AB''C''$, being each similar to ABC , the one by hypothesis, and the other by the preceding Theorem, are similar to each other, and as we have, by construction, $AB'' = A'B'$, these triangles must be equal, since the ratio of similitude is unity (D. I.). Now, since $B''C''$ is parallel to BC , the angles of the triangle $AB''C''$ are respectively equal to the angles of the triangle ABC ; that is to say, we have the angle at A common, $B'' = B$, $C'' = C$. Then, also, we must have $A' = A$, $B' = B$, $C' = C$.

We see that the *equal angles* are opposite the *homologous sides*.

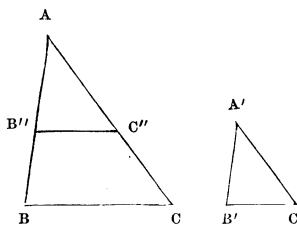
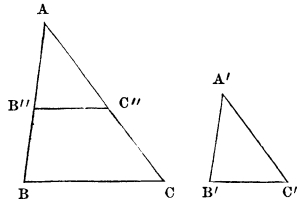
THEOREM VI.

Two triangles, which have the angles of the one respectively equal to the angles of the other, are similar, and the sides opposite the equal angles are homologous.

In the two triangles ABC , $A'B'C'$, suppose $A = A'$, $B = B'$, $C = C'$.

As in the preceding Theorem, take $AB'' = A'B'$ and draw $B''C''$ parallel to BC . The two triangles $A'B'C'$, $AB''C''$, are equal, having a side equal, $A'B' = AB''$, and the adjacent angles equal, $A' = A$, $B' = B = B''$. Now the triangle $AB''C''$ is similar to the triangle ABC (T. IV.); hence its equal $A'B'C'$ will be similar to ABC . Consequently we have this series of equal ratios:

$$AB : A'B' :: AC : A'C' :: BC : B'C'.$$



The *homologous* sides, AB and A'B', AC and A'C', BC and B'C', as we see, are *opposite* the equal angles $C' = C$, $B' = B$, $A' = A$.

THEOREM VII.

Two triangles are similar, when the sides of the one are respectively parallel or perpendicular to the sides of the other, and the homologous sides are the parallel or the perpendicular sides.

From the preceding Theorem we see that the demonstration will be made out if we can show that these triangles have their angles respectively equal.

Let AB and A'B', AC and A'C', BC and B'C', represent the respective sides, either parallel or perpendicular, we are to show that $C = C'$, $B = B'$, $A = A'$.

In effect, we know (B. I., T. III., C.) that the angles C and C', B and B', A and A', must be either equal or supplementary.

Now, in the first place, it is impossible that three set of angles should be supplementary, since we should then have

$$A + A' + B + B' + C + C' = 6 \text{ right angles,}$$

which is absurd (B. I., T. V., C. I.).

Neither can two set of these angles be supplementary; as, for example, A and A', B and B', for then we should have

$$A + A' + B + B' = 4 \text{ right angles,}$$

which is also absurd.

It is then necessary that two angles, at least, of the first triangle should be equal respectively to two angles of the second; consequently, the third angle of the first triangle must equal the third angle of the second (B. I., T. V., C. III.). So that we shall have

$$A = A', B = B', C = C';$$

and consequently the triangles are similar, and we have this series of equal ratios (T. VI.):

$$BC : B'C' :: AC : A'C' :: AB : A'B'.$$

From the method of demonstration we see that the *homologous* sides BC and B'C', AC and A'C', AB and A'B', are the *parallel* or the *perpendicular* sides.

THEOREM VIII.

Two triangles, which have an angle of the one equal to an angle of the other, and the sides containing these angles proportional, are similar.

In the two triangles ABC , $A'B'C'$, suppose we have $A = A'$, and

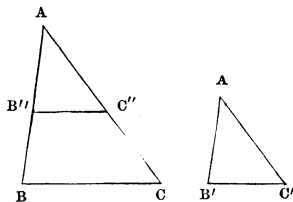
$$AB : A'B' :: AC : A'C'.$$

Take on AB and AC , $AB'' = A'B'$, $AC'' = A'C'$, and draw $B''C''$. The two triangles $A'B'C'$, $AB''C''$, are equal, having an equal angle included between equal sides each to each. Now, by hypothesis, we have

$$AB : A'B' :: AC : A'C',$$

consequently, $AB : AB'' :: AC : AC''$;

and $B''C''$ is parallel to BC (T. III., Cor. II.), and the triangle $AB''C''$ is therefore similar to ABC (T. IV.). Then, also, is $A'B'C'$ similar to ABC .

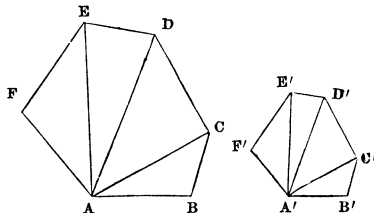


SIMILAR POLYGONS.

THEOREM IX.

Two similar polygons have their homologous sides proportional, and the homologous angles respectively equal.

For, by the similarity of the triangles ABC and $A'B'C'$, ACD and $A'C'D'$, ADE and $A'D'E'$, etc., we deduce the following series of equal ratios:



$$AB : A'B' :: AC : A'C' :: BC : B'C' ::, \text{ etc.}$$

$$AC : A'C' :: AD : A'D' :: CD : C'D' ::, \text{ etc.}$$

$$AD : A'D' :: AE : A'E' :: DE : D'E' ::, \text{ etc.}$$

&c.

&c.

&c.

Omitting the ratios which are common in these series, we have

$$AB : A'B' :: BC : B'C' :: CD : C'D' :: DE : D'E' ::, \text{ etc.}$$

Again, from the similarity of these same triangles, it follows that their homologous angles are equal (T. V.), and consequently the respective angles of the two polygons are equal, since they are composed of angles equal each to each. Thus, $A = A'$, $B = B'$, $C = C'$, $D = D'$, etc.

THEOREM X.

Two polygons are similar when they have their sides, taken in the same order, proportional, and their angles, also taken in the same order, equal each to each.

That is to say, when we have

$$AB : A'B' :: BC : B'C' :: CD : C'D' ::, \text{ etc.,}$$

and $A = A'$, $B = B'$, $C = C'$, $D = D'$, etc.

Let the two polygons be decomposed into triangles by drawing lines from A and A'.

The two triangles ABC, A'B'C', are similar, having equal angles, $B = B'$, included between proportional sides AB, A'B', BC, B'C' (T. VIII.). Hence, angle $ACB = \text{angle } A'C'B'$, and

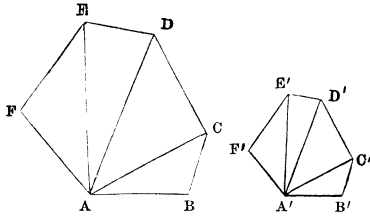
$$AC : A'C' :: BC : B'C'.$$

But, by hypothesis, we have

$$BC : B'C' :: CD : C'D'.$$

Consequently, $AC : A'C' :: CD : C'D'.$

Now, comparing the two triangles ACD, A'C'D', we see that the angles ACD, A'C'D' are equal, being the differences between the equal angles BCD, B'C'D', and ACB, A'C'B', and we have just shown that the sides about these equal angles are proportional, hence these triangles are also similar. The same may be shown for all the couples of triangles, ADE and A'D'E', AEF and A'E'F'. Hence the polygons are similar (D. III.).



Scholium. Two parallelograms are similar, when they have an angle of the one equal to an angle of the other, and the sides containing the equal angles proportional.

Two rhombuses are similar, when they have an angle of the one equal to an angle of the other.

All squares are similar figures.

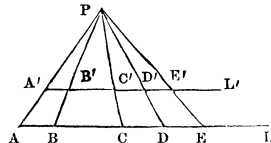
All regular polygons of the same number of sides are similar figures.

PROPORTIONAL LINES.—PROPERTIES OF THE SIDES OF TRIANGLES.

THEOREM XI.

The intercepted portions of two parallel lines, made by any number of lines proceeding from the same point, are proportional.

The comparison of the different couples of similar triangles PAB and PA'B', PBC and PB'C', etc., immediately gives the following series of equal ratios :



$$\begin{aligned} PA : PA' :: AB : A'B' :: PB : PB', \\ PB : PB' :: BC : B'C' :: PC : PC', \\ PC : PC' :: CD : C'D' :: PD : PD', \\ \quad \&c. \qquad \quad \&c. \qquad \quad \&c. \end{aligned}$$

Now, all the ratios are equal, however far they may be extended, since the last in each line above is the first of the following line. Then, if we use only the intermediate ratios, we shall have

$$AB : A'B' :: BC : B'C' :: CD : C'D' :: DE : D'E' ::, \text{ etc.,}$$

which establishes the proposition.

When the point P is within the parallels, the same demonstration will apply. In this case, the corresponding segments of the two lines AL and A'L' will be situated in opposite directions.

THEOREM XII.

If a line be drawn bisecting either angle of a triangle, it will divide the opposite side into two segments proportional to their adjacent sides.

If the angle BAC is bisected by AD, we shall have

$$AB : AC :: BD : CD.$$

Draw BE and CF perpendicular to AD. The two triangles ABE, ACF are similar, since they are equiangular; hence

$$AB : AC :: BE : CF.$$

But the two triangles BDE, CFD are also similar, and give

$$BE : CF :: BD : CD.$$

Consequently, by equality of ratios, we have

$$AB : AC :: BD : CD.$$

Cor. The line AD', bisecting the supplementary angle at A, that is to say, the line perpendicular to AD, will also determine, on the prolongation of the side BC, two segments BD', CD', such as to give the proportion,

$$AB : AC :: BD' : CD'.$$

For, drawing BE' and CF' perpendicular to AD', we have, by reason of similar triangles,

$$AB : AC :: BE' : CF'; \quad BE' : CF' :: BD' : CD';$$

consequently, $AB : AC :: BD' : CD'.$

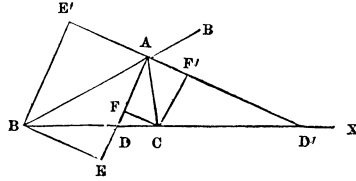
Scholium. From the two proportions,

$$\begin{aligned} AB : AC &:: BD : CD, \\ AB : AC &:: BD' : CD', \end{aligned}$$

just demonstrated, we deduce the following:

$$BD : CD :: BD' : CD';$$

and the straight line BC is said to be divided *harmonically* at the points D and D'.



THEOREM XIII.

If from the right angle of a right-angled triangle a line is drawn perpendicular to the hypotenuse :

I. *The two partial triangles thus formed will be similar to the whole triangle, and consequently similar to each other.*

II. *The perpendicular will be a mean proportional between the two segments of the hypotenuse.*

III. *Either side containing the right angle will be a mean proportional between its adjacent segment and the hypotenuse.*

First. The two triangles ABC, ABD are similar, since they have the common angle at B, and the angle BAC of the one equal to the angle BDA of the other, each being a right angle.

For the same reason, ACB is similar to ACD, consequently these triangles are similar to each other.

Secondly. Comparing the two partial triangles ABD, ACD, we have

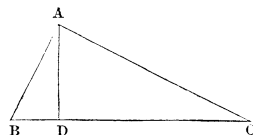
$$BD : AD :: AD : DC.$$

Thirdly. Comparing the total triangle ABC with its partial triangle ABD, observing that in these triangles BC and AB are homologous, being the hypotenuses, that AB and BD are also homologous, being opposite the equal angles BCA, BAD, we shall obtain the proportion,

$$BC : AB :: AB : BD.$$

The comparison of the triangles ABC, ACD will give, in like manner,

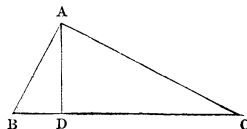
$$BC : AC :: AC : DC.$$



THEOREM XIV.

In any right-angled triangle, the square, or second power, of the numerical value of the hypotenuse, is equal to the sum of the squares of the numerical values of the other two sides.

For, taking the product of the means and extremes of the last two proportions of the preceding Theorem, we have



$$AB^2 = BC \times BD; \quad AC^2 = BC \times DC;$$

from which, by adding member to member,

$$AB^2 + AC^2 = BC \times (BD + DC) = BC \times BC;$$

that is, $AB^2 + AC^2 = BC^2$.

Cor. When $AB = AC$, we have $BC^2 = 2AB^2$; that is, in any right-angled isosceles triangle, the square of the hypotenuse is double the square of one of the sides. Consequently, the second power of the numerical value of the diagonal of a square is double that of the side. And the diagonal and side must be to each other in the ratio of $\sqrt{2}$ to 1; that is, they are incommensurable, as has already been shown (B. II., P. XIV.).

Scholium I. From the two equations,

$$AB^2 = BC \times BD, \quad AC^2 = BC \times DC,$$

we deduce the proportion,

$$AB^2 : AC^2 :: BC \times BD : BC \times DC, \text{ or, } :: BD : DC.$$

In a similar manner, by combining the identical equation $BC^2 = BC^2$, with the same two equations above, we have

$$BC^2 : AB^2 :: BC \times BC : BC \times BD, \text{ or, } :: BC : BD;$$

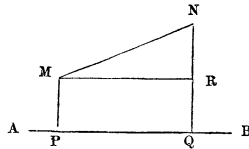
and $BC^2 : AC^2 :: BC \times BC : BC \times DC, \text{ or, } :: BC : DC.$

From this we see that, in any right-angled triangle, the squares or second powers of the sides containing the right angle and of the hypotenuse, are proportional to the segments of this hypotenuse, and the hypotenuse itself. That is to say, we have

$$AB^2 : AC^2 : BC^2 :: BD : DC : BC.$$

Scholium II. The segments BD , DC , formed by the perpendicular AD on the hypotenuse BC , are called the *projections* of the two sides on the hypotenuse.

In general, the *projection* of any definite straight line MN , on another indefinite straight line AB , is the distance PQ of the intersection of the perpendiculars drawn through the extremities of the first line to the second.



If through M we draw MR parallel to PQ , we shall have $MR = PQ$, and the right-angled triangle MNR will give

$$MN^2 = MR^2 + NR^2 = PQ^2 + (NQ - MP)^2.$$

That is—*The square of a straight line of determinate length is equal to the square of its projection on another line, plus the square of the difference of the perpendiculars which determine this projection.*

THEOREM XV.

In any obtuse-angled triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, increased by twice the product of either of the sides containing the obtuse angle into the projection of the other side on the prolongation of the first.

If the triangle ABC is obtuse-angled at A, we shall have

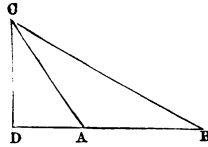
$$BC^2 = AB^2 + AC^2 + 2AB \times AD.$$

For, since $BD = AB + AD$, if we square both members of this equation, using the algebraic formula, we shall have

$$BD^2 = AB^2 + AD^2 + 2AB \times AD.$$

Adding now, to each member of this, CD^2 , and observing that $BD^2 + CD^2 = BC^2$, and also that $AD^2 + CD^2 = AC^2$, we shall obtain

$$BC^2 = AB^2 + AC^2 + 2AB \times AD.$$



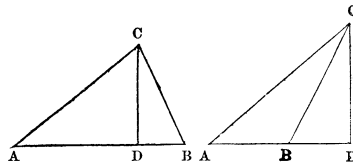
THEOREM XVI.

In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, diminished by twice the product of one of these sides, by the projection of the other on the preceding one, produced if necessary.

If the angle A is acute, we shall have

$$BC^2 = AB^2 + AC^2 - 2AB \times AD.$$

For, $BD = AB - AD$, or $AD - AB$, according to which diagram we use; but in both cases BD is the difference of AB



and AD; consequently, if we take the squares of both members, by the well-known algebraic formula, we shall have

$$BD^2 = AB^2 + AD^2 - 2AB \times AD.$$

Adding to each member of this DC^2 , and observing that $BD^2 + CD^2 = BC^2$, and also that $AD^2 + DC^2 = AC^2$, we shall obtain

$$BC^2 = AB^2 + AC^2 - 2AB \times AD.$$

This result will hold good even when the perpendicular CD coincides with the side CB, in which case $AD = AB$, and the above will become $BC^2 = AC^2 - AB^2$, or, $AC^2 = AB^2 + BC^2$, which we know to be true, since the triangle ABC is, in this case, right-angled at B (T. XIV.).

Scholium. A triangle is *right-angled*, *acute-angled*, or *obtuse-angled*, according as the square of the longest side is *equal*, *less*, or *greater* than the sum of the squares of the other two sides.

If a triangle has 3, 4, and 5 for the numerical values of its sides, it will be right-angled, since

$$5^2 = 3^2 + 4^2.$$

One having 4, 5, and 6 for its sides, is acute-angled, since

$$6^2 < 4^2 + 5^2.$$

One having 2, 3, and 5 for its sides, is obtuse-angled, since

$$5^2 > 2^2 + 3^2.$$

Thus we are furnished with a second very simple method for determining the kind of triangle, when its sides are given (B. I., T. XXXV., S.).

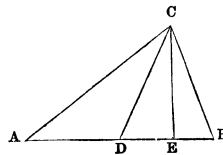
THEOREM XVII.

In any triangle, the sum of the squares of any two sides is equal to twice the square of half the third side, increased by twice the square of the line drawn from the middle of this third side to the opposite angle.

If CD is drawn bisecting AB, we shall have

$$AC^2 + BC^2 = 2AD^2 + 2CD^2.$$

For, the two triangles ADC, CDB, the



one obtuse-angled and the other acute-angled at D, give, by the two preceding Theorems,

$$AC^2 = AD^2 + CD^2 + 2AD \times DE,$$

$$BC^2 = BD^2 + CD^2 - 2DB \times DE;$$

adding these, and observing that $AD = DB$, we have

$$AC^2 + BC^2 = 2AD^2 + 2CD^2.$$

Cor. The sum of the squares of the four sides of any parallelogram is equal to the sum of the squares of its diagonals.

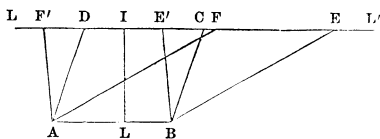
This proposition is easily deduced from the above.

DETERMINATION OF AREAS.

THEOREM XVIII

Two parallelograms having equal bases and the same altitude, are equivalent.

We will suppose the one figure placed upon the other, so that their inferior bases, which are equal, may coincide. Let $ABCD$ be the first parallelogram, and $ABEF$ or $ABE'F'$ the second. Since these parallelograms have the same altitude, their superior bases will be situated in the same indefinite line LL' , parallel to the lower base.



If now, we confine our attention to the two parallelograms $ABCD$, $ABEF$, we see that the triangles BCE , ADF are equal, having equal angles CBE and DAF (B. I., T. III.) included between equal sides, namely, $BE = AF$, $BC = AD$ (B. I., T. XXXI.). If, now, from the quadrilateral $ABED$ we subtract the triangle BEC , we shall have left the parallelogram $ABCD$; but if from the same quadrilateral we subtract the equal triangle ADF , we shall have left the parallelogram $ABEF$. Hence the two parallelograms are equivalent.

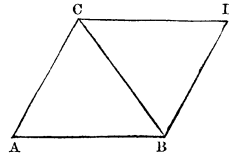
We may in the same way prove that the parallelograms $ABCD$, $ABE'F'$ are equivalent.

Cor. A parallelogram is equivalent to a rectangle having the same base and the same altitude. Two rectangles of the same base and the same altitude are *equal*, and consequently *equivalent*.

THEOREM XIX.

A triangle is equivalent to half of a parallelogram having the same base and the same altitude.

For, through B and C, draw BD parallel to AC, and CD parallel to AB; we thus form a parallelogram having the same base and the same altitude as the triangle.



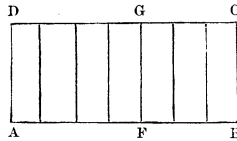
The triangle is one half this parallelogram (B. I., T. XXXI., Cor I.).

Cor. Two triangles of the same base, or of equal bases, and the same altitude, are equivalent.

THEOREM XX.

Two rectangles of the same altitude are to each other as their bases.

Let ABCD, AFGD be the two rectangles having the common altitude AD; then will they be to each other as their bases AB, AF.



We will first suppose the bases AB, AF to be commensurable; as, for example, suppose they are to each other as 7 to 4. If we divide AB into seven equal parts, AF will contain four of these parts; at each point of division draw lines perpendicular to the base, thus forming seven partial rectangles, which will be equal, since they have equal bases and the same altitude (T. XVIII.).

Now, as the rectangle ABCD contains seven of these equal rectangles, while AFGD contains only four, it follows, that

$$ABCD : AFGD :: 7 : 4;$$

but

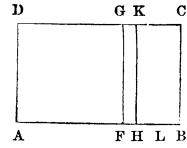
$$AB : AF :: 7 : 4;$$

therefore,

$$ABCD : AFGD :: AB : AF.$$

We will suppose, in the second place, that the bases AB , AF are incommensurable, still we shall have

$$ABCD : AFGD :: AB : AF.$$



For, if this proportion is not true, the first three terms remaining the same, the fourth term will be either greater or less than AF . Suppose it to be greater, and that we have

$$ABCD : AFGD :: AB : AL.$$

Divide the line AB into equal parts, each of which shall be less than FL . There will be at least one point of division, as at H , between F and L . Through this point H draw the perpendicular HK , then will the bases AB and AH be commensurable and we shall have, by the first part of this proposition,

$$ABCD : AHKD :: AB : AH.$$

But, by supposition,

$$ABCD : AFGD :: AB : AL.$$

Since, in these two proportionals, the antecedents are equal, the consequents are proportional, and we have

$$AHKD : AFGD :: AH : AL.$$

But AL is greater than AH ; hence, if this proportion is correct, we must have $AFGD$ greater than $AHKD$; on the contrary, it is less, hence the above proportion is impossible. Therefore, $ABCD$ cannot be to $AFGD$ as AB is to a line greater than AF .

By similar reasoning, we can show that the fourth term of the proportion cannot be smaller than AF .

Hence, whatever may be the ratio of the bases, the two rectangles $ABCD$, $AFGD$, of the same altitude, will be to each other as their bases AB , AF .

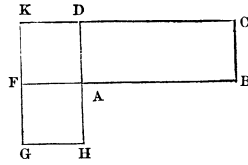
THEOREM XXI.

Any two rectangles are to each other as the product of their bases multiplied by their altitudes.

Let $ABCD$, $AFGH$ be two rectangles, then will

$$ABCD : AFGH :: AB \times AD : AH \times AF.$$

Having placed the two rectangles so that the angles at A may be vertical and opposite, produce the sides CD, GF until they meet in K; the two rectangles ABCD, AFKD, having the same altitude AD, are to each other as



their bases AB, AF. In like manner, the two rectangles AFKD, AFGH, having the same altitude AF, are to each other as their bases, AD, AF. Hence we have these two proportions :

$$ABCD : AFKD :: AB : AF.$$

$$AFKD : AFGH :: AD : AF.$$

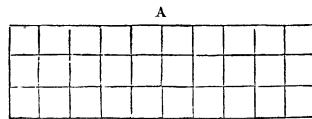
Multiplying the corresponding terms of these proportions together, and observing to omit AFKD, since it will occur in an antecedent and consequent, we shall have

$$ABCD : AFGH :: AB \times AD : AF \times AH.$$

Scholium. Hence we are at liberty to take for the measure of a rectangle, the product of its base by its altitude, provided we understand by this product the same as the product of two numbers, which numbers denote the linear units in the base and altitude respectively.

This measure, however, is not absolute, but only relative. It supposes that the area of any other rectangle is estimated in a similar manner, by measuring its sides by the same linear unit; we shall thus obtain a second product, and the ratio of these two products is the same as that of the two rectangles, in accordance with this proposition.

For example, if the base of the rectangle A is ten units, and its height three, the rectangle will be represented by the number $10 \times$

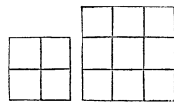


$3 = 30$, a number which, thus isolated, has no signification; but if we have a second rectangle B, whose base is twelve units and height seven, this second rectangle will be represented by the number $12 \times 7 = 84$. From which we conclude that the two rectangles A and B are to each other as 30 to 84. If we take the rectangle A as the unit of measure of surfaces, the rectangle B will have for its measure $\frac{84}{30}$; that is, it will be $\frac{14}{5}$ of our superficial units.

It is more common and more simple to take a square for the unit of surface, and we choose a square whose side is a unit of length. In this case, the measure which we have regarded as relative becomes absolute; for example, the number 30, which measured the rectangle A, represents 30 superficial units, or 30 squares, each side of which is a unit long.

In geometry, we frequently confound the product of two lines with that of their *rectangle*, and this expression is even employed in arithmetic to denote the product of two unequal numbers; but we use the term *square* to denote the product of a number multiplied by itself.

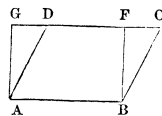
The squares of the numbers 1, 2, 3, etc., are 1, 4, 9, etc. From which we see that the square formed on a line of double length is quadruple; on a line of triple length it is nine times as great, and so of other squares.



THEOREM XXII.

The area of any parallelogram is equal the product of its base by its height.

For the parallelogram ABCD is equivalent to the rectangle ABFG, which has the same base AB, and same altitude BF (T. XVIII., C.). But the rectangle ABFG is measured by $AB \times BF$, consequently the parallelogram is measured by $AB \times BF$.



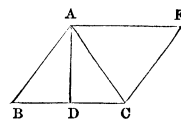
Cor. Parallelograms of the same base are to each other as their altitudes, and parallelograms of the same altitude are to each other as their bases; and in general, parallelograms are to each other as the products of their bases multiplied by their altitudes.

THEOREM XXIII.

The area of a triangle is equal to the product of its base by half its altitude.

For, the triangle ABC is half the parallelogram ABCF (T. XIX.).

The parallelogram is measured by the base BC multiplied into the altitude AD; therefore,

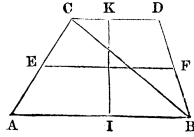


the triangle is measured by the base BC into half the altitude AD.

Cor. I. Triangles of the same altitude are to each other as their bases; and triangles of the same or equal bases are to each other as their altitudes.

Cor. II. The area of a trapezoid ABDC is equal to the product of half the sum of the bases, AB, CD, into the altitude IK; or, it is equal to the product IK into the straight line EF drawn at equal distance between the bases.

In effect this trapezoid is composed of two triangles CAB, BCD; and by the Theorem, we have



$$ABC = \frac{AB \times IK}{2}, \quad BCD = \frac{CD \times IK}{2};$$

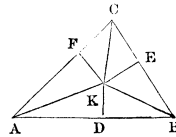
$$\text{then } ABDC = \frac{AB \times IK}{2} + \frac{CD \times IK}{2} = \frac{AB + CD}{2} \times IK.$$

And as we have proved (B. I., T. XXXIII.) that $EF = \frac{AB + CD}{2}$, we have $ABDC = IK \times EF$.

THEOREM XXIV.

The area of a triangle is equal to half the product of its perimeter by the radius of its inscribed circle.

Let K be the centre of the inscribed circle. Draw KD, KE, KF respectively perpendicular to the sides AB, BC, AC. Also draw KA, KB, KC.



We obviously have

$$ABC = AKB + BKC + AKC.$$

But (T. XXIII.) we have

$$AKB = \frac{AB \times KD}{2}, \quad BKC = \frac{BC \times KE}{2}, \quad AKC = \frac{AC \times KF}{2}.$$

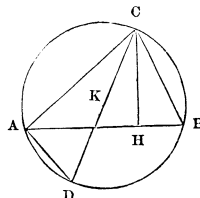
And since $KD = KE = KF$, we have

$$ABC = \frac{(AB + BC + AC) \times KD}{2}.$$

THEOREM XXV.

The area of a triangle is equal to the product of the three sides divided by twice the diameter of its circumscribed circle.

Let K be the centre of the circumscribed circle. Draw the diameter CKD, and the chord AD, and the perpendicular CH, which will be the altitude of the triangle.



The two triangles CAD, CHB are right-angled, the one at A (B. II., T. X., Cor. I.), and the other at H; moreover, the angles at D and B are equal, being inscribed in the same segment ADBC. Hence these triangles are similar, and their homologous sides being proportional, we have

$$CD : CB :: AC : CH.$$

Consequently,
$$CH = \frac{CB \times AC}{CD}.$$

But we also have
$$ABC = \frac{AB \times CH}{2},$$

in which, substituting the value of CH just found, we have

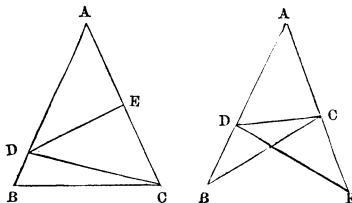
$$ABC = \frac{AB \times CB \times AC}{2CD}.$$

RATIOS BETWEEN THE AREAS OF SIMILAR FIGURES

THEOREM XXVI.

The areas of two triangles which have an equal angle, are proportional to the rectangles of the sides containing the equal angle.

Referring to either figure, let the two triangles ABC, ADE have the angle at A common; then we shall have



$$ABC : ADE :: AB \times AC : AD \times AE.$$

For, joining D and C, we have, since triangles of the same altitude are to each other as their bases,

$$\begin{aligned} ABC : ADC &:: AB : AD, \\ ADC : ADE &:: AC : AE. \end{aligned}$$

Multiplying together the corresponding terms of these proportions, and omitting the common term ADC which enters into the antecedent and consequent of the first couplet, we have

$$ABC : ADE :: AB \times AC : AD \times AE.$$

Cor. Whenever the side DE of the second triangle is parallel to BC of the first, the triangle ADC is a mean proportional between the triangles ABC and ADE.

In effect we have, in this case (T. III.),

$$AB : AD :: AC : AE;$$

so that the two first proportions of the preceding Theorem have equal ratios, and give

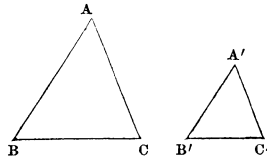
$$ABC : ADC :: ADC : ADE.$$

THEOREM XXVII.

The areas of similar triangles are proportional to the squares of their homologous sides.

These triangles being similar, are equiangular (T. V.).

Hence, by the preceding Theorem, we have



$$ABC : A'B'C' :: AB \times AC : A'B' \times A'C' ;$$

but since the triangles are similar, we also have

$$AC : A'C' :: AB : A'B'.$$

Multiplying the corresponding terms of these two proportions, omitting the common factor AC which enters into the antecedents, and the common factor A'C' which enters into the consequents, we have

$$ABC : A'B'C' :: AB^2 : A'B'^2.$$

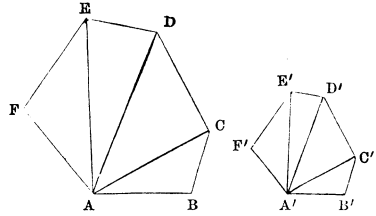
THEOREM XXVIII.

The perimeters of two similar polygons are proportional to their homologous sides. And their areas are proportional to the squares of these sides.

First. By reason of the similarity of these polygons, we have (T. IX.),

$$AB : A'B' :: BC : B'C' :: CD : C'D' ::, \text{ etc.}$$

Now the sum of these antecedents, $AB + BC + CD +$, etc., which makes the perimeter of the first polygon, is to the sum of their consequents $A'B' + B'C' + C'D' +$, etc., which makes the perimeter



of the second polygon, as any one antecedent is to its corresponding consequent, and therefore as AB is to $A'B'$.

Secondly. Since these polygons are similar, they are each composed of the same number of similar triangles, which, compared together, are each to each, in the same ratio, that of the squares of the homologous sides. It follows, then, that the sum of the triangles which compose the first polygon, is to the sum of the triangles which compose the second polygon, as the squares of the homologous sides of the two polygons. That is, we shall have

$$ABCDEF : A'B'C'D'E'F' :: AB^2 : A'B'^2.$$

Cor. *The perimeters of similar figures are proportional to their homologous lines, and their areas are proportional to the squares of those lines.*

For, by the definition of similar figures, the ratio of any two homologous lines, which is called the ratio of similitude, is constantly the same.

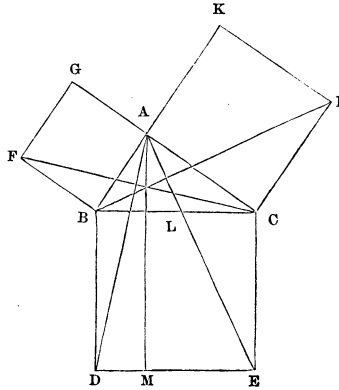
COMPARISON OF SQUARES CONSTRUCTED ON CERTAIN LINES.

THEOREM XXIX.

The square constructed on the hypotenuse of a right-angled triangle, is equivalent to the sum of the squares constructed respectively on the other two sides.

This Theorem is not a fundamental one, like Theorem XIV., but its importance and its fecundity have given rise to several demonstrations founded solely upon the *equality* and *equivalence* of figures. We shall confine ourselves to the one generally given in works of geometry.

The three squares being constructed without the triangle ABC, we draw from the right angle at A the line AL perpendicular to the hypotenuse, and produce it to meet DE at M, thus dividing the square BCED into the two rectangles BDML, LCEM. We also draw AD and CF.



Now, the angles ABD, FBC are equal, being each composed of a right angle and of the common angle ABC; the lines AB and BF are equal, being sides of the same square, so also are BC and BD equal, for a like reason. Hence the two triangles BAD, BFC are equal (B. I., T. XX.). But the triangle ABD is half the rectangle BLMD, since they have the same base BD and the same altitude BL (T. XIX.). For the same reason the triangle BFC is half the square ABFG, for they have the same base BF and the same altitude AB; consequently,

$$BLMD = ABFG.$$

We can prove in the same manner, by drawing AE and BL, that

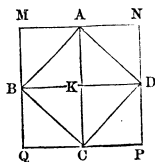
$$LMEC = ACIH;$$

and as the square BDEC is equivalent to BLMD + LMEC, there will result

$$\text{square BDEC} = \text{square ABFG} + \text{square ACIK}.$$

Cor. I. The square MNPQ, constructed on the diagonal BD or AC of a square ABCD, is double the square itself.

This is obvious from the simple inspection of the figure. In effect, the four squares AKBM, AKDN, BKCQ, DKCP, are equal, and respectively the doubles of the triangles AKB, AKD, BKC, DKC, of which the sum is equal to the square ABCD.



Cor. II. We have already shown that the squares ABFG, ACIK (see first figure) are respectively equivalent to the rectangles BLMD, LMEC. And since these rectangles and the square BCED have the common altitude BD, they are to each other as their bases BL, LC, and BC (T. XX.). Hence we obtain this relation :

$$AB^2 : AC^2 : BC^2 :: BL : LC : BC.$$

Cor. III. Since the two squares ABFG, ACIK are respectively equivalent to the rectangles BLMD, LMEC, we have

$$AB^2 = BC \times BL, \quad AC^2 = BC \times LC;$$

from which we deduce these two proportions :

$$BC : AB :: AB : BL, \quad BC : AC :: AC : LC.$$

These two proportions, together with the relations of the preceding Corollary, correspond in all respects with the third portion of Theorem XIII., and Scholium I. of Theorem XIV. We have thus, by another method, established properties already demonstrated.

Cor. IV. In short, if on the three sides of a right-angled triangle ABC, we suppose three similar polygons to be constructed, as these polygons will be proportional to the squares of their homologous sides (T. XXVIII.), and we have this relation between these sides :

$$BC^2 = AB^2 + AC^2;$$

it follows, that, of the three similar polygons, the polygon constructed on the hypotenuse is equivalent to the sum of the polygons constructed on the sides containing the right angle.

Scholium. The algebraic formulas,

$$(p+q)^2 = p^2 + q^2 + 2pq, \quad (p-q)^2 = p^2 + q^2 - 2pq,$$

which have been made the bases of Theorems XV. and XVI., and the other well-known formula,

$$(p+q) \times (p-q) = p^2 - q^2,$$

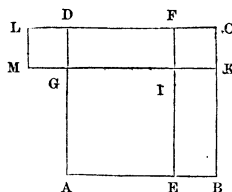
being translated into geometrical language, will give rise to some new Theorems in regard to areas, with which we will close this Book.

THEOREM XXX.

The square constructed on the sum or on the difference of two lines, is equivalent to the sum of the squares constructed respectively on these lines, plus or minus twice their rectangle.

The simple inspection of the figure is nearly sufficient to satisfy one of the truth of this double proposition.

First. Let AE be the greater of the two lines, EB the less. Construct the squares AEIG, ABCD, and produce EI until it meets CD at F.



The square constructed on AB, the sum of the two lines, is evidently composed of the squares AEIG, IKCF, constructed on the line AE and on the line IK = EB, increased by the two rectangles BEIK, GFD, which have for their respective bases GI = AE, EI = AE, and for altitudes GD = EB, IK = EB.

Hence, we have

square AB = square AE + square EB + twice rectangle AE × EB.

Secondly. Let AB be the greater line, BE the less, that which gives AE for the difference of these two lines. Construct the same figure as in the first case, and, in addition, construct the square GDLM equal to IKCF.

The square AEIG is equal to the square ABCD plus the square GDLM, minus the two rectangles EBCF, MIFL. Now, these two rectangles have respectively for bases FE = AB, IM = GK = AB, and for altitudes BE = IK = IF.

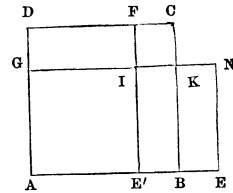
Hence we have

square AE = square AB + square EB - twice rectangle AB × BE.

THEOREM XXXI.

The rectangle constructed on the sum and the difference of two lines, is equivalent to the difference of the squares constructed on these two lines.

Let AB be the greater line, $BE = BE'$ the less, so that AE will represent the sum of these two lines, and AE' their difference. Construct on AE as a base, and $AG = AE'$ as an altitude, the rectangle $AENG$, also the square $ABCD$, and draw $E'I$ perpendicular to AB , forming the square $AE'IG$.



The two rectangles $BENK$, $GIFD$ are equal, having equal bases and equal altitudes, namely, $BK = AG = GI = AE'$, $EB = E'B = IK = IF = DG$; hence it follows that the rectangle $AENG$ is equivalent to the figure $DFIKBA$. But this figure is the difference of the squares constructed on AB and on $IK = BE' = BE$.

Hence, we have

$$\text{rectangle } AENG = \text{square } AB - \text{square } BE.$$

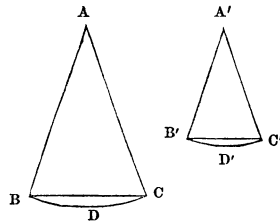
FOURTH BOOK.

THE PROPORTIONS OF LINES AND THE AREAS OF FIGURES IN CONNECTION WITH THE CIRCLE.

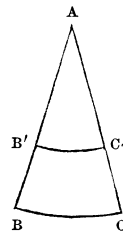
DEFINITIONS.

I. *Similar arcs, Sectors, or Segments*, are those which, in different circles, correspond to equal angles at the centre.

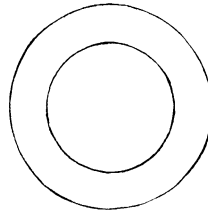
Thus, if the angles at A and A' are equal, the arcs BC, B'C' will be similar, the sectors BAC, B'A'C' will be similar, so also will the segments BDC, B'D'C'.



II. When two similar sectors are superposed so that their equal angles coincide, their difference is called a *circular trapezoid*. Thus the space BB'C'C is a circular trapezoid.



III. The space included between two concentric circumferences is called a *circular ring*.



IV. The arc of a circle is said to be *rectified*, when it is *developed*; that is, unfolded or drawn out into a straight line. It is only in this rectified form that we can conceive of its length, since it could not otherwise be compared with the linear unit.

PROPORTIONAL LINES CONNECTED WITH THE CIRCLE.

THEOREM I.

When two chords intersect each other within a circle, the segments will be reciprocally or inversely proportional.

Let the two chords AA' , BB' intersect at P . Drawing the auxiliary chords AB' , $A'B$, we thus obtain two triangles PAB' , $PA'B$, which are similar, since the angles at P are equal (B. I., T. I.); the angles at A and B are equal, so also are the angles at A' and B' (B. II., T. X.), hence these triangles are equiangular, and consequently similar (B. III., T. VI.), and their homologous sides give this proportion,

$$PA : PB :: PB' : PA'.$$

Scholium I. When one of the chords is a diameter, and the other is perpendicular to it, we have

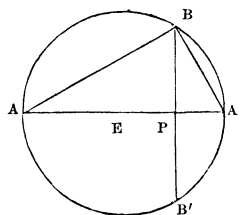
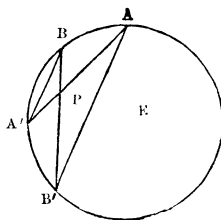
$$PA : PB :: PB : PA'.$$

The line PB , drawn perpendicular to a diameter, is called an *ordinate* to this diameter. Hence, we have this condition:

In a circle any ordinate to a diameter is a mean proportional between the two segments of this diameter.

We can show that this property is one of the consequences of the properties of right-angled triangles; because if we join A , A' with the point B , we shall form a right-angled triangle (B. II., T. X., C. I.), which gives (B. III., T. XIII.) this proportion,

$$PA : PB :: PB : PA'.$$



Scholium II. This same right-angled triangle ABA' gives the proportion,

$$AA' : AB :: AB : AP.$$

That is,

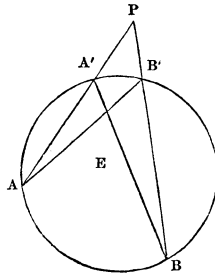
Any chord which is drawn through the extremity of a diameter is a mean proportional between its projection (B. III., T. XIV., S. II.) on the diameter and the diameter itself.

THEOREM II.

When two secants intersect each other, without a circle, they will be reciprocally proportional to their external segments.

Let the two secants intersect each other at P. Draw, as in the preceding Theorem, the chords AB' , $A'B$, and the two triangles $AB'P$, $BA'P$, will be similar, since the angle at P is common, and $\angle A = \angle B$, each being measured by half of the arc $A'B'$ (B. II., T. X.), and the homologous sides give the proportion

$$PA : PB :: PB' : PA'.$$



THEOREM III.

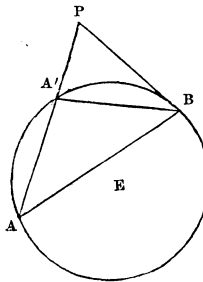
When a secant and tangent are drawn from the same point, the tangent is a mean proportional between the secant and its external segment.

This Theorem is in reality only a particular case of the preceding, when the points B, B' of the secant PB, are united in one point.

But we will give a direct demonstration, by drawing the chords AB, A'B. We have, in effect, two triangles, PAB, PA'B, similar, since the angle at P is common and the two angles at A and B are equal (B. II., T. X., C. II.).

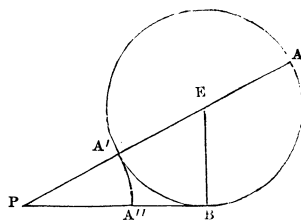
Hence, we have this proportion,

$$PA : PB :: PB : PA'.$$



Scholium I. As a particular case, let us suppose the secant to pass through the centre of the circle, and that the tangent is equal to the diameter; that is, $PB = AA'$ the proportion will then become

$$PA : AA' :: AA' : PA'.$$



When a straight line, as PA , is thus divided at a point A' , so that the greater segment AA' , is a mean proportional between the whole line and the other segment, it is said to be *divided into mean and extreme ratio*.

The above proportion will give

$$AA' : PA - AA' :: PA' : AA' - PA',$$

or,

$$AA' : PA' :: PA' : AA' - PA'.$$

Now if we take PA'' equal to PA' , and observe that $AA' = PB$, and consequently that $AA' - PA' = PB - PA'' = A''B$, we shall have

$$PB : PA'' :: PA'' : A''B.$$

From this we see that the straight line PB is also divided at the point A'' *into mean and extreme ratio*.

Scholium II. The three preceding Theorems give immediately these two equations,

$$PA \times PA' = PB \times PB', \quad PA \times PA' = PB^2,$$

which may be included in one single proposition, as follows:

The product of the distances from the same point, either within or without a circle, to two points of its circumference, taken in the same straight line, is always the same.

In the case of the tangent we must regard the two points of the circumference as united.

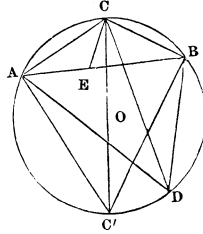
THEOREM IV.

In a quadrilateral inscribed in a circle, the rectangle of the diagonals is equal to the sum of the rectangles of the opposite sides.

Let ACBD be the inscribed quadrilateral, and we shall have

$$AB \times CD = AC \times BD + AD \times BC.$$

For, drawing CE, making the angle ACE equal to the angle BCD, and consequently the angle ECB equal to the angle ACD.



The two triangles CEA, CBD are similar, since the angle ACE equals the angle DCB, by construction, and CAE = CDB, since each is in the same segment (B. II., T. X.). Their homologous sides give this proportion,

$$AC : CD :: AE : BD ;$$

consequently, $CD \times AE = AC \times BD.$

The comparison of the homologous sides of the two triangles BEC, DAC, which are similar, gives

$$AD : BE :: CD : BC,$$

and $CD \times BE = AD \times BC.$

Adding the corresponding members of these equations we obtain

$$CD \times (AE + BE) = AC \times BD + AD \times BC,$$

or $AB \times CD = AC \times BD + AD \times BC.$

Scholium. This Theorem has many important applications.

FIRST. To find the chord AB of the sum of two arcs AC, CB, when their chords are known.

For abridgment, let a, b, c denote the three sides BC, AC, AB of the triangle ABC, and R the radius of its circumscribing circle. Draw the diameter COC' = 2R, also the supplementary chords AC', BC'.

Now the inscribed quadrilateral ACBC' will give

$$AB \times CC' = AC \times BC' + AC' \times CB.$$

But the triangles CAC', CBC' being right-angled, since they are each in a semicircle, give

$$AC' = \sqrt{CC'^2 - AC^2}, \quad BC' = \sqrt{CC'^2 - BC^2};$$

and since $CC' = 2R$, $AC = b$, $BC = a$, the foregoing expressions will become

$$AC' = \sqrt{4R^2 - b^2}, \quad BC' = \sqrt{4R^2 - a^2},$$

these values, substituted in the first condition, cause it to become

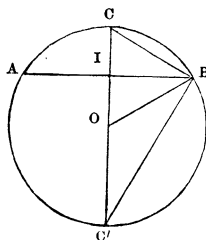
$$c \times 2R = b \sqrt{4R^2 - a^2} + a \sqrt{4R^2 - b^2};$$

hence,
$$c = \frac{b}{2R} \sqrt{4R^2 - a^2} + \frac{a}{2R} \sqrt{4R^2 - b^2}. \quad (1)$$

SECONDLY. To find the chord AB = c, of double an arc, when we know the chord BC = a of the arc itself.

This will evidently be accomplished by making $b = a$ in the general expression just formed; so that, in this case, we have

$$c = \frac{a}{R} \sqrt{4R^2 - a^2}. \quad (2)$$



THIRDLY. To find the chord of half an arc, when we know the chord of the arc itself.

In the triangle CBC', $CB^2 = CI \times CC'$ (B. III., T. XIII.).

Now, $CI = OC - OI = OC - \sqrt{OB^2 - BI^2}$ (B. III., T. XIV.);

or,
$$CI = R - \sqrt{R^2 - \frac{1}{4}c^2}.$$

Hence,
$$CB^2 = a^2 = 2R \times (R - \sqrt{R^2 - \frac{1}{4}c^2}) = 2R^2 - R\sqrt{4R^2 - c^2};$$

or,
$$a = \sqrt{2R^2 - R\sqrt{4R^2 - c^2}}. \quad (3)$$

It would not be difficult to deduce, algebraically, the value of a in terms of c and b , from formula (1); that is, to find the chord of the difference of two arcs, when we know their chords.

When found, the expression is

$$a = \frac{c}{2R} \sqrt{4R^2 - b^2} - \frac{b}{2R} \sqrt{4R^2 - c^2}. \quad (4)$$

We would also remark that formula (3) could have been obtained very readily from (2) by the ordinary rules of algebra.

DETERMINATION OF THE SIDES AND OF THE AREAS OF
REGULAR POLYGONS.

THEOREM V.

The area of any regular polygon is equal to half the product of its perimeter by its apothem.

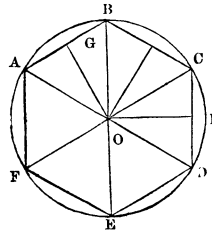
For, the equal isosceles triangles OAB, OBC, OCD, etc., give

$$OAB = AB \times \frac{1}{2} \text{ of } OG,$$

$$OBC = BC \times \frac{1}{2} \text{ of } OG,$$

$$OCD = CD \times \frac{1}{2} \text{ of } OG,$$

$$\&c., \quad \&c.$$



Consequently,

$$\text{area } ABCDEF = (AB + BC + CD + \text{etc.}) \times \frac{1}{2} OG;$$

or, more concisely, $A = \frac{1}{2} p \times r.$

A denoting the area of the polygon, p its perimeter, and r the apothem, or the radius of the inscribed circle.

Scholium. Calling R the radius of the polygon, n the number of sides, and a the length of one of the sides, we have, from the right-angled triangle OGA,

$$OG = \sqrt{OA^2 - AG^2}; \text{ that is, } r = \sqrt{R^2 - \frac{1}{4}a^2}.$$

We also have $p = na;$

consequently,
$$A = \frac{na \sqrt{4R^2 - a^2}}{4}.$$

THEOREM VI.

The perimeters of two regular polygons of the same number of sides are proportional to the radii of their inscribed circles, or of their circumscribed circles; and their areas are proportional to the squares of those radii.

Since these polygons are regular, and have the same number of sides, they are similar figures. Consequently, the isosceles triangles which have their vertices at the centre, and for bases

the sides of the polygons, are each to each equiangular and similar. The radii of the inscribed circles and of the circumscribed circles are homologous lines. Therefore the proposition is true (B. III., T. XXVIII., C.).

Let R and R' , r and r' , a and a' , p and p' , A and A' , denote, respectively, the radii, the apothems, the sides, the perimeters, and the areas of these two polygons, and we shall have as follows:

$$\begin{aligned} p : p' :: R : R' :: r : r' :: a : a', \\ A : A' :: R^2 : R'^2 :: r^2 : r'^2 :: a^2 : a'^2. \end{aligned}$$

THEOREM VII.

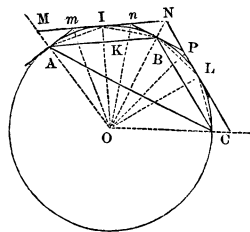
In a given circle, knowing the side of a regular inscribed polygon, we can obtain:

I. *The value of the side of a regular inscribed polygon of half the number of sides.*

II. *The value of the side of a regular inscribed polygon of double the number of sides.*

III. *The radius and the side of a circumscribed polygon similar to the given polygon.*

Using the same diagram as under Prob. XI., Book Second, let us represent the side AB , of the given polygon, by a , its radius OA by R .



First. Since AC subtends an arc double the arc subtended by $AB = a$, it is sufficient to substitute AC for c in formula (2), scholium to T. IV. Which thus becomes

$$AC = \frac{a}{R} \sqrt{4R^2 - a^2}. \quad (1)$$

Secondly. We obtain, in the same way, the value of AI , by means of formula (3), by substituting AI for a , and a for c . Making these substitutions we have

$$AI = \sqrt{2R^2 - R \sqrt{4R^2 - a^2}}. \quad (2)$$

Thirdly. As to the values of OM and MN , the two similar

triangles OMN, OAB, of which OI and OK are homologous lines, give

$$OM : OA :: OI : OK; \text{ hence, } OM = \frac{OA \times OI}{OK},$$

$$MN : AB :: OI : OK; \text{ hence, } MN = \frac{AB \times OI}{OK};$$

but we have $OA = OI = R$, $AB = a$, and

$$OK = \sqrt{OA^2 - AK^2} = \sqrt{R^2 - \frac{1}{4}a^2} = \frac{1}{2}\sqrt{4R^2 - a^2}.$$

Therefore,

$$OM = \frac{2R^2}{\sqrt{4R^2 - a^2}}, \quad MN = \frac{2a \times R}{\sqrt{4R^2 - a^2}}. \quad (3)$$

Scholium. Knowing the radius and the side of each of the three new polygons, we are able to deduce the values of their areas by substituting, in the formula (T. V., S.),

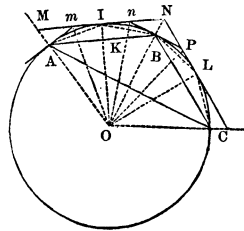
$$A = \frac{na \sqrt{4R^2 - a^2}}{4}$$

for n , a , R , the respective values just obtained. $\frac{n}{2}$ must be used for n , for the first polygon, and $2n$ for n , for the second. It remains the same for the third.

THEOREM VIII.

The area of a regular inscribed polygon, and that of a regular circumscribed one of the same number of sides being known, we are always able to obtain the areas of two regular polygons, the one inscribed and the other circumscribed, of twice the number of sides.

Let A and B denote respectively the areas of the inscribed and circumscribed polygons, each having a number of sides denoted by n , and A' and B' , the areas respectively of the inscribed and circumscribed polygons of double the number of sides.



By inspecting the diagram we see that we have the following relations: $A = 2n \times OAK$, $B = 2n \times OMI$, $A' = 2n \times OAI$, $B' = 2n \times OAmI$; hence, the ratios between the areas A , B , A' , B' , taken two and two, are the same as those between the triangles OAK , OMI , OAI , and of the quadrilateral $OAmI$; thus the whole is reduced to the determination of the ratios which exists between these last four figures.

This being premised, we have immediately, in comparing the three triangles OMI , OAI , OAK , and observing that AK is parallel to MI ,

$$OMI : OAI :: OAI : OAK \quad (\text{B. III., T. XXVI., C.});$$

or, replacing these ratios of triangles by those of the polygons, we have

$$B : A' :: A' : A;$$

that is,

$$A' = \sqrt{A \times B}. \quad (1)$$

In the second place, since in the triangle OMI , the straight line Om divides the angle O into two equal parts, we have the proportion

$$Mm : mI :: OM : OI \quad (\text{B. III., T. XII.}).$$

Now, the two triangles OMm , OmI , having the same altitude OI , give

$$OMm : OmI :: Mm : mI \quad (\text{B. III., T. XXIII., C. I.}).$$

Also, since the triangles OAI , OAK have the same altitude AK , we have

$$OAI : OAK :: OI : OK :: OM : OA :: OM : OI ;$$

hence, we have

$$OMm : OmI :: OAI : OAK ;$$

consequently,

$$OMm + OmI : OmI :: OAI + OAK : OAK ;$$

or doubling the consequents, and observing that $OMm + OmI = OMI$, and $2 \times OmI = OAmI$, we have

$$OMI : OAmI :: OAI + OAK : 2 \times OAK.$$

Finally, substituting, in place of the three triangles OMI ,

OAI, OAK, and of the quadrilateral OAmI, the polygons of which they are like parts, we shall obtain

$$B : B' :: A + A' : 2A ;$$

consequently, $B' = \frac{2 \times A \times B}{A + A'}$, or $B' = \frac{2 \times A \times B}{A + \sqrt{A \times B}}$. (2)

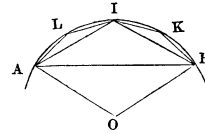
Scholium. These two formulas show, at a glance, that $A' > A$ and $B' < B$.

Thus the areas of the regular inscribed polygons in the same circle will continue to augment as the number of sides, arising from doubling each time, increases. But the areas of the circumscribed polygons will decrease as the sides increase.

However great may be the number of sides of the regular inscribed and circumscribed polygons, it is evident that the area of an inscribed polygon cannot exceed the area of the circle, and the area of a circumscribed polygon cannot be less than the circle.

The same is true in regard to their perimeters as compared with the circumference of the circle.

To prove this, it will be necessary only to consider the portion of the figure corresponding to the arc AB, since the same course of reasoning may be repeated for each of the corresponding portions BC, CD, etc.



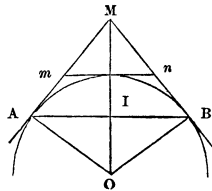
Let the arc AB be divided into two, four, eight, etc., equal parts. Draw the chords of these new arcs.

From the definition of a straight line, we evidently have

chord $AB < AI + IB < AL + LI + IK + KB < \text{etc.}, < \text{arc } AB$.

Hence, the perimeters of the regular inscribed polygons, in the same circle, will continue to augment as the number of sides increase, without being able to surpass the circumference of the circle.

Now, let AM and BM be two half sides of the first regular circumscribed polygon corresponding to the arc AB; mn, and Am, Bn a side and half sides of the regular circumscribed polygon of double the number of sides.



We shall have $mM + Mn > mn$; adding $Am + nB$ to each member, we have

$$Am + mM + Mn + nB, \text{ or } AM + MB > Am + mn + nB.$$

Hence, the perimeters of the regular circumscribed polygons will continue to decrease as the number of sides increase, without being able to become less than the circumference.

This being premised, we give, in mathematics, the name of **LIMIT** to a constant and determinate magnitude, towards which a variable magnitude is constantly converging, either by increasing or diminishing. Hence, we say that:

The area of a circle is the SUPERIOR limit of the areas of the regular inscribed polygons, and the INFERIOR limit of the areas of the regular circumscribed polygons.

The circumference of a circle is the superior limit of the perimeters of the first polygons, and the inferior limit of the second.

We feel authorized to regard the circle as a regular polygon of an infinite number of sides, infinitely small; these sides are called the *elements* of the curve.

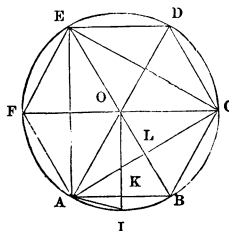
And this new definition of a circle, if it does not at first appear very rigorous, has at least the advantage of giving greater simplicity and precision to our demonstrations.

THE DETERMINATION OF THE SIDES AND OF THE AREAS OF REGULAR POLYGONS OF A PARTICULAR KIND.

THEOREM IX.

The side of a regular inscribed hexagon is equal to the radius.

Draw OA and OB ; the value of the angle O , of the triangle OAB , is $\frac{4}{6} = \frac{2}{3}$ of a right angle, hence the sum of the other two angles is $2 - \frac{2}{3} = \frac{4}{3}$ of a right angle, and since $OA = OB$, it follows that the angle $OAB = \text{angle } OBA = \frac{1}{2}$ of $\frac{4}{3} = \frac{2}{3}$ of



a right angle; hence the triangle OAB is equilateral, and gives $AB = OA = OB = R$.

Cor. I. The side AC of an inscribed equilateral triangle is to the radius as $\sqrt{3} : 1$.

If in formula (2), of Scholium to T. IV., we make $a = R$, we shall find

$$AC = \frac{R}{R} \sqrt{3R^2} = R\sqrt{3};$$

hence,

$$AC : R :: \sqrt{3} : 1.$$

Cor. II. The side of a circumscribed equilateral triangle is double the side of the inscribed one.

To show this, it is sufficient to substitute $R\sqrt{3}$ for a in formula (3) of T. VII., which thus becomes

$$MN = \frac{2R^2\sqrt{3}}{\sqrt{4R^2 - 3R^2}} = 2R\sqrt{3}.$$

We may remark, that the altitude EL of the inscribed triangle is equal to $\frac{3}{2}R$. For, we have $EL = EO + OL = R + \frac{1}{2}R$, since the figure OABC is a lozenge, consequently $EL = \frac{3}{2}R$.

It follows, then, that the altitude of the circumscribed triangle is equal to $3R$.

Cor. III. If, in formula (2) of T. VII., we substitute R for a , we shall find

$$AI = (2R^2 - R\sqrt{4R^2 - R^2})^{\frac{1}{2}} = R\sqrt{2 - \sqrt{3}},$$

for the side of a regular inscribed dodecagon.

Cor. IV. Finally, in formula (3), T. VII., substituting R for a , we find the radius and a side of a regular circumscribed hexagon, as follows:

$$OM = \frac{2}{3}R\sqrt{3}; \quad MN = \frac{2}{3}R\sqrt{3}.$$

Scholium. We are now prepared, by the aid of the general formula (T. V.),

$$A = \frac{na\sqrt{4R^2 - a^2}}{4},$$

to calculate the areas of these different polygons.

Thus, for example, if we make $n = 6$, $a = R$, we shall obtain for the expression of the area of the regular inscribed hexagon,

$$A = \frac{3}{2}R^2\sqrt{3}.$$

By making $n = 3$, $a = R\sqrt{3}$, we find

$$A = \frac{3}{4}R^2\sqrt{3},$$

for the area of the inscribed equilateral triangle.

In a similar manner we proceed for the other cases.

THEOREM X.

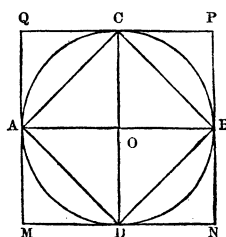
The side of the inscribed square is to the radius in the ratio of $\sqrt{2}$ to 1.

Draw the two diameters AB, CD perpendicular to each other, and draw the chords AC, CB, BD, DA. The figure ADBC is evidently a square; and we have

$$AC^2 = AO^2 + OC^2 = 2R^2;$$

consequently $AC = R\sqrt{2}$, and

$$AC : R :: \sqrt{2} : 1.$$



Scholium. In drawing through the points A, D, B, C, tangents, we form the circumscribed square; and we have

$$MN = AB = 2R.$$

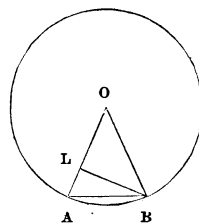
That is—*The side of the circumscribed square is equal to the diameter of the circle.*

The areas of these two polygons are expressed respectively by $2R^2$ and $4R^2$.

THEOREM XI.

The side of a regular inscribed decagon is equal to the greater segment of the radius, when divided into mean and extreme ratio.

Draw the radii OA, OB to the extremities of a side AB. The angle O of the triangle OAB is equal to $\frac{4}{10} = \frac{2}{5}$ of a right angle; there remains, then, $2 - \frac{2}{5} = \frac{8}{5}$ of a right angle, for the sum of the other two angles; and as $OA = OB$, it follows that the angle $OAB = OBA = \frac{4}{5}$ of a right angle.



Thus, each of the angles at the base of the triangle OAB is double the angle at the vertex.

This being premised, draw BL bisecting the angle OBA, and we shall have this proportion,

$$AL : LO :: AB : OB \quad (\text{B. III., T. XII.}).$$

But since the angle LBO = LBA = LOB = $\frac{2}{5}$, and consequently ALB = 2LOB = $\frac{4}{5}$, the two triangles OLB, ALB are also isosceles, and give OL = LB = AB, and also having OB = OA, the above proportion becomes

$$AL : OL :: OL : OA.$$

From which we see that the point L divides the radius OA into mean and extreme ratio (T. III., S. I.), and that the greater segment OL is equal to AB a side of the regular decagon.

Scholium I. We will deduce the numerical value of this side algebraically. Denoting it by x , we have (T. III., S. I.),

$$R : x :: x : R - x,$$

$$\text{or, } x^2 + Rx = R^2 \text{ and } x = \frac{R}{2}(\pm \sqrt{5} - 1).$$

That is, the numerical value of the side of a regular inscribed decagon is $\frac{R}{2}(\sqrt{5} - 1)$.

Scholium II. If for a in formula (2) of Scholium to T. IV. we substitute $\frac{R}{2}(\sqrt{5} - 1)$, which we have just found for the value of a side of a regular inscribed decagon, we shall find $\frac{R}{2}(10 - 2\sqrt{5})^{\frac{1}{2}}$ for the side of a regular inscribed pentagon.

Now, since $\left[\frac{R}{2}(10 - 2\sqrt{5})^{\frac{1}{2}}\right]^2 = \left[\frac{R}{2}(\sqrt{5} - 1)\right]^2 + R^2$, it follows, that *the square of the side of a regular inscribed pentagon is equal to the square of the side of the regular inscribed decagon increased by the square of the radius.*

THEOREM XII.

The side of a regular inscribed pentadecagon is the chord of the difference between the arcs subtended respectively by the sides of a hexagon and a decagon.

For, we have $\frac{1}{6} - \frac{1}{10} = \frac{5}{30} - \frac{3}{30} = \frac{2}{30} = \frac{1}{15}$; which shows that—*The difference between the arcs subtended by the sides of a hexagon and of the decagon is equal to one fifteenth of the entire circumference.*

If we wish the numerical value of the side of the pentadecagon, we may employ formula (4) of T. IV., in which, for c , we must substitute the value of the side of a hexagon, and for b we must use the side of a decagon.

General Scholium. It results from what has been said in the last four Theorems, and of the principles before established, that we have geometrical methods for constructing regular inscribed and circumscribed polygons of 3, 6, 12, 24, etc., of 4, 8, 16, 32, etc., of 5, 10, 20, 40, etc., in short, of 15, 30, 60, etc., number of sides.

That we also have arithmetical methods for calculating the sides, and afterwards the areas, of all these polygons, if not exactly—since nearly all the numbers which enter into our calculations are incommensurable—yet to as close a degree of approximation as may be desired.

MEASURE OF THE CIRCLE AND OF ITS CIRCUMFERENCE.

THEOREM XIII.

The area of a circle is equal to half the product of the circumference into its radius.

In effect, let us consider a series of regular circumscribed polygons, each having double the number of sides of the preceding one. The superficies of these polygons have for their respective measures half the product of each perimeter into its apothem—that is, into the radius of the circle. Hence, the area of the circle, which is the *limit* (T. VIII., S.) of these polygons,

has for its measure half the product of the circumference, which is the *limit* of the perimeters, into its radius.

Or, considering the circle as a regular polygon of an infinite number of sides, it immediately follows that its area is equal to half the product of its perimeter, that is, of its circumference, into its radius.

Scholium I. Let R , C , and S denote respectively the radius, circumference, and surface of any circle. We shall have

$$S = \frac{1}{2}C \times R = C \times \frac{1}{2}R.$$

We must keep in mind that S is an *abstract number*, expressing the ratio of the surface of the circle to the unit of surface; C and R are ratios of the circumference and of the radius to the linear unit.

Scholium II. We may also say that the area of a circle is equal to that of a triangle having for its base the *rectified* circumference (D. IV.), and for its altitude the radius of the circle (B. III., T. XXIII.).

THEOREM XIV.

In two circles, the circumferences are proportional to their radii or to their diameters, and their areas are proportional to the squares of their radii.

First. Let us conceive a series of regular polygons, each having double the number of sides of the preceding one, to circumscribe the circumference C ; also another series of similar polygons to circumscribe the circumference C' ; and let us designate by R , R' the constant apothems of these two series of polygons; by p , p' the perimeters of two similar polygons, taken, the one in the first series, the other in the second.

This being supposed, we shall have this proportion,

$$p : p' :: R : R',$$

which is applicable to any two similar polygons whatever; it must therefore hold good at the *limit* of the perimeters, that is, when p , p' become C , C' ; hence,

$$C : C' :: R : R', \text{ or, } :: 2R : 2R'.$$

Secondly. Multiplying this last proportion by the obvious proportion,

$$\frac{1}{2}R : \frac{1}{2}R' :: R : R', \text{ or, } :: 2R : 2R',$$

it becomes, $C \times \frac{1}{2}R : C' \times \frac{1}{2}R' :: R^2 : R'^2$, or $4R^2 : 4R'^2$.

Or, since (T. XIII., S. I.) $C \times \frac{1}{2}R = S$, $C' \times \frac{1}{2}R' = S'$, this becomes

$$S : S' :: R^2 : R'^2, \text{ or } 4R^2 : 4R'^2.$$

Or, considering the circle as a regular polygon of an infinite number of sides, the above follows immediately from T. XXVIII., B. III.

Cor. The proportion $C : C' :: 2R : 2R'$, may be written

$$C : 2R :: C' : 2R',$$

and being applicable to any number whatever of circumferences, we have

$$C : 2R :: C' : 2R' :: C'' : 2R'' :: C''' : 2R''' ::, \text{ etc.,}$$

from which we infer that

The ratio of the circumference to the diameter is a constant number.

We usually denote by π this constant ratio.

Scholium. If R , C , and S denote respectively the radius, the circumference, and the surface of any circle whatever, we shall have this proportion

$$\pi : 1 :: C : 2R;$$

consequently, $C = 2\pi R$, from which, by multiplying each term by $\frac{1}{2}R$, we find $C \times \frac{1}{2}R = S = \pi R^2$.

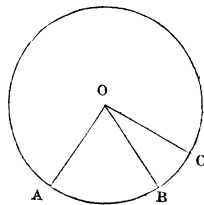
From this, we see that the circumference of any circle may be found by multiplying its diameter ($2R$) by π ; that its surface may be obtained by multiplying the square of its radius by π .

THEOREM XV.

The area of a circular sector is equal to half the product of its arc into its radius.

In effect, from the definition of a circular sector, it follows that in the same circle two sectors are proportional to their angles, and consequently to their corresponding arcs.

Hence, comparing the given sector AOB



with the sector AOC, which has its angle AOC right, we have this proportion,

$$\text{sector AOB} : \text{sector AOC} :: \text{arc AB} : \text{AC};$$

or, multiplying the two consequents by four, and denoting by C, S, the circumference and surface, or area of the circle, we have

$$\text{sector AOB} : S :: \text{arc AB} : C.$$

Again, multiplying the terms of the second couplet of this last proportion by $\frac{1}{2}R$, we obtain,

$$\text{sector AOB} : S :: \text{arc AB} \times \frac{1}{2}R : C \times \frac{1}{2}R.$$

But, $C \times \frac{1}{2}R = S$ (T. XIII., S. I.); consequently,

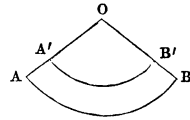
$$\text{sector AOB} = \text{arc AB} \times \frac{1}{2}R = \frac{1}{2} \text{arc AB} \times R.$$

Scholium. The area of a sector is equal to that of a triangle having for its base the *rectified arc* of the sector, and for its altitude the radius of the circle.

THEOREM XVI.

The areas of two similar sectors are proportional to the squares of their radii, or of their arcs.

The sectors being similar, have their angles equal. We will, therefore, suppose the one placed upon the other so that their angles may coincide.



Since (T. XV.) $\text{sector AOB} = \text{arc AB} \times \frac{1}{2}OA$, $\text{sector A'OB'} = \text{arc A'B'} \times \frac{1}{2}OA'$, we have

$$\text{sector AOB} : \text{sector A'OB'} :: \text{arc AB} \times \frac{1}{2}OA : \text{arc A'B'} \times \frac{1}{2}OA';$$

but the arcs AB, A'B' being evidently in the same ratio as the entire circumferences of which they are parts, which circumferences are in the ratio of their radii OA, OA' (T. XIV.), there will result

$$\text{arc AB} : \text{arc A'B'} :: OA : OA',$$

$$\text{hence, } \text{arc AB} \times \frac{1}{2}OA : \text{arc A'B'} \times \frac{1}{2}OA' :: OA^2 : OA'^2;$$

$$\text{consequently, } \text{sector AOB} : \text{sector A'OB'} :: OA^2 : OA'^2,$$

$$\text{also, } \text{sector AOB} : \text{sector A'OB'} :: (\text{arc AB})^2 : (\text{arc A'B'})^2.$$

Scholium I. The difference $AA'B'B$ between the similar sectors OAB , $OA'B'$, which is called a *circular trapezoid* (D. II.), has for its measure the product of half the sum of its bases into the difference of their radii.

To prove this, draw the indefinite tangents BL , $B'L'$, and take on BL the portion BK equal to the rectified arc BA , and draw KO meeting $B'L'$ in K' . Then, by reason of the similarity of figures, we shall have

$$OB : OB' :: BK : B'K';$$

$$OB : OB' :: \text{arc } BA : \text{arc } B'A';$$

consequently,

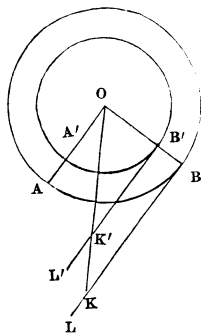
$$BK : B'K' :: \text{arc } BA : \text{arc } B'A'.$$

But $BK = \text{arc } BA$ by construction, therefore $B'K' = \text{arc } B'A'$. Hence the two sectors OAB , $OA'B'$ are respectively equivalent to the two triangles OBK , $OB'K'$ (T. XV., S.); it therefore follows that the circular trapezoid is equivalent to the right-angled trapezoid $KBB'K'$, which has for its measure $\frac{BK + B'K'}{2} \times BB'$ (B. III., T. XXIII., C. II.). Consequently we have

$$\text{circular trapezoid } AA'B'B = \frac{\text{arc } AB + \text{arc } A'B'}{2} \times BB'.$$

It would be, moreover, easy to prove that the half sum of the bases, or the mean difference between the two bases, is a concentric arc with AB , $A'B'$, at equidistance from the same.

Scholium II. The *circular ring* (D. III.), that is to say, the space between the two concentric circumferences (see last diagram), is only a particular case of the circular trapezoid, consequently its area is expressed by the product of the mean circumference into its width BB' , the difference of the radii.



RATIO OF THE CIRCUMFERENCE TO THE DIAMETER.

FIRST METHOD.

We know that the side of a regular inscribed hexagon is equal to the radius (T. IX.). If we suppose, then, the radius $R = 1$, we can find the side of a regular inscribed polygon of twelve sides, by making $a = R = 1$, in formula (2) of Theorem VII., that which gives $\sqrt{2 - \sqrt{3}}$ for a side of a regular inscribed dodecagon. Its perimeter is $12\sqrt{2 - \sqrt{3}}$; consequently the ratio of this perimeter to the diameter is $6\sqrt{2 - \sqrt{3}}$.

Were we to substitute anew, in the same formula, 1 for R , and $\sqrt{2 - \sqrt{3}}$ for a , we should find a side of an inscribed regular polygon of twenty-four sides. Thus we might continue to find the sides of the successive polygons, each having double as many sides as the preceding one, to any extent we please.

As the number of sides of the polygons increase, their perimeters will approximate more nearly to the circumference of the circle; we are thus enabled, by this method, to obtain the value of π to as close a degree of approximation as we please.

SECOND METHOD.

We have demonstrated (T. XIV., S.) that the area of a circle which has R for its radius, is equal to πR^2 .

Now, if we make $R = 1$ in this expression, it will become π , which shows that

The ratio of the circumference to the diameter is equal to the area of a circle whose radius is unity.

This being supposed, let us start with an inscribed and circumscribed square, whose areas are respectively 2 and 4 (T. X., S.). And the formulas

$$A' = \sqrt{A \times B}, \quad B' = \frac{2 \times A \times B}{A + A'}$$

(T. VIII.), if we suppose $A = 2$, $B = 4$, will become

$$A' = \sqrt{8}, \quad B' = \frac{16}{2 + \sqrt{8}} = 4(\sqrt{8} - 2),$$

which, converted into decimals, will give 2.8284271, etc.,

3.3137085, etc., for the areas, respectively, of the inscribed and circumscribed octagons.

Using these values in the same formulas, we shall obtain the areas of the inscribed and circumscribed polygons of sixteen sides, which in turn will make known the areas of polygons of thirty-two sides, and so on, till we arrive at an inscribed and circumscribed polygon, differing from each other, and consequently from the circle, so little that either may be considered as equivalent to it. The subjoined table exhibits the *area*, or numerical expression for the surface of each succeeding polygon, carried to seven places of decimals.

No. of sides.	Area of the inscribed polygon.	Area of the circumscribed polygon.
4	2.0000000	4.0000000
8	2.8284271	3.3137085
16	3.0614674	3.1825979
32	3.1214451	3.1517249
64	3.1365485	3.1441184
128	3.1403311	3.1422236
256	3.1412772	3.1417504
512	3.1415138	3.1416321
1024	3.1415729	3.1416025
2048	3.1415877	3.1415951
4096	3.1415914	3.1415933
8192	3.1415923	3.1415928

It appears, then, that the inscribed and circumscribed polygons of 8192 sides differ so little from each other, that the numerical value of each, as far as six places of decimals, is absolutely the same; and as the circle is between the two, it cannot, strictly speaking, differ from either so much as they do from each other; so that the number 3.141592 expresses the area of a circle whose radius is 1: that is, the value of π to six places of decimals is 3.141592.

THIRD METHOD.

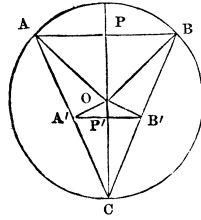
Instead of seeking the approximate value of the circumference, or of the area of a circle, of which the radius is 1, *we shall seek the value of the radius corresponding to a given circumference.*

For this purpose, we will premise the following

LEMMA.

Having given the radius R and the apothem r of a regular polygon, it is required to find the radius R' and the apothem r' of a second regular polygon having the same perimeter as the first and double the number of sides.

Suppose the given polygon to be circumscribed, and that AB is one of the sides, AOB the angle at the centre, $OA = R =$ the radius, and $OP = r =$ the apothem.



This being supposed, the angle at the centre of the second polygon is half the angle AOB . If we produce PO until it meets the circumference at C , and draw the chords CA , CB , the angle ACB , half of AOB is the angle at the centre of the second polygon.

Also, if we draw OA' perpendicular to CA , and draw $A'B'$ parallel to AB , we shall have $A'B' = \frac{1}{2}AB$ (B. I., T. XXXIII., Cor.) for the side of the second polygon, and CA' will be its radius, and CP' its apothem.

The similar triangles CAP , $CA'P'$, give

$$CP' = \frac{1}{2}CP = \frac{1}{2}(CO + OP) = \frac{1}{2}(OA + OP),$$

consequently, $r' = \frac{1}{2}(R + r)$.

Again, the right-angled triangle $OA'C$ gives

$$CA'^2 = CO \times CP' = OA \times CP';$$

that is, $R' = \sqrt{R \times r'}$; and as r' has already been found, the problem is resolved.

These formulas are much more simple than those already used in the first and second methods.

Now, to make an application of these formulas, we will take for our given polygon a square whose side is 1, and consequently its perimeter is 4. Its radius is $\frac{1}{2}\sqrt{2}$, being obviously half its diagonal, and its apothem is $\frac{1}{2}$.

If, then, in the formula $r' = \frac{1}{2}(R + r)$, we make $R = \frac{1}{2}\sqrt{2} = 0.7071068$, etc., $r = \frac{1}{2} = 0.5$, we shall find $r' = 0.6035534$, etc. Formula $R' = \sqrt{R \times r'}$ will now give $R' = 0.6532815$, etc. We

thus find the radius and apothem of a regular polygon of eight sides, and whose perimeter is 4.

Using these values in the same formulas, they will in turn make known the radius and apothem of a regular polygon of sixteen sides, having the same perimeter, 4. We give the results of these successive operations, to seven places of decimals, in the following table :

No. of sides.	Apothems.	Radii.
4	0.5000000	0.7071068
8	0.6035534	0.6532815
16	0.6284174	0.6407289
32	0.6345731	0.6376435
64	0.6361083	0.6368754
128	0.6364919	0.6366836
256	0.6365878	0.6366357
512	0.6366117	0.6366237
1024	0.6366177	0.6366207
2048	0.6366192	0.6366199
4096	0.6366195	0.6366197
8192	0.6366196	0.6366196

From this table, we see that a circle whose circumference is 4 has for its radius 0.6366196, etc., and consequently for its diameter 1.2732392, etc. Hence, the ratio of this circumference to its diameter, or the value of π , is

$$\frac{4}{1.2732392, \text{ etc.}} = 3.1415931, \text{ etc.}$$

This value of π differs less than a unit in the sixth decimal place from the true value.

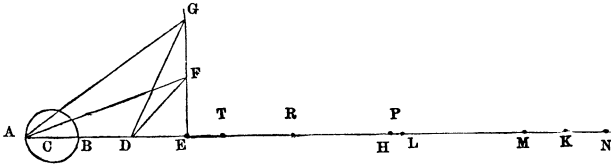
There are other methods, depending upon a knowledge of the higher branches of mathematics, by which this value of π has been extended to more than two hundred decimal places. The value to thirty-one decimals is

$$\pi = 3.1415926535897932384626433832795.$$

The following *original geometrical construction* is very simple, and gives this ratio sufficiently accurate for all practical purposes :

Let ACB be the diameter of the given circle : produce it towards N ; take BD and DE each equal to AB ; through E draw

EG perpendicular to AE, and take EF and FG each equal to AB; join AG, AF, DG, and DF. Set off on the line EN, from E, the distances EH and HK, each equal to AG; then set off, in the opposite direction, the distance KL equal to AF, and from L set off LM equal to DG; also set off MN equal to DF. Then bisect EN at the point P; bisect EP at the point R, and, finally, trisect ER at the point T; then will CT be the circumference of the circle, nearly.



For, by construction, we have, if we call the diameter a unit, $OE = 2\frac{1}{2}$; $EL = 2EH - KL = 2\sqrt{13} - \sqrt{10}$; $LM = \sqrt{5}$; $MN = \sqrt{2}$. Therefore,

$$EN = 2\sqrt{13} - \sqrt{10} + \sqrt{5} + \sqrt{2};$$

and $ET = \frac{1}{12}(2\sqrt{13} - \sqrt{10} + \sqrt{5} + \sqrt{2});$

and, therefore,

$$CT = 2\frac{1}{2} + \frac{1}{12}(2\sqrt{13} - \sqrt{10} + \sqrt{5} + \sqrt{2}) = 3.1415922, \text{ etc.},$$

which is the ratio true to six decimals. For simplicity and accuracy, a better graphic method of finding this ratio can hardly be expected, or even desired.

PROBLEMS,

WHICH REFER TO THE THIRD AND FOURTH BOOKS.

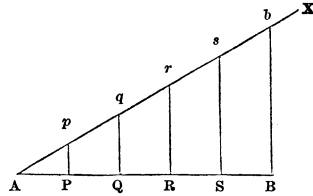
CONSTRUCTION OF PROPORTIONAL LINES.

PROBLEM I.

To divide a given line into any number of equal parts.

To make a definite case, suppose we wish to divide AB into five equal parts.

Through A draw any indefinite line AX, making an angle with AB. Lay off on this line any convenient length five times, as Ap, pq, qr, rs, sb. Join the last extremity b with B, and through the other points of division draw parallel to Bb, lines cutting AB in the points P, Q, R, and S (B. II., P. VI.).



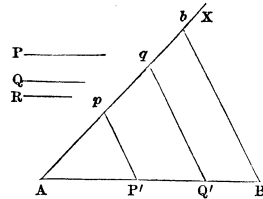
The straight line will thus be divided into five equal parts (B. III., T. I.).

PROBLEM II.

To divide a given line into parts proportional to given lines.

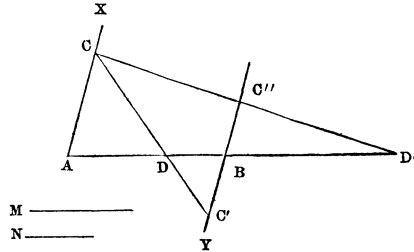
As a definite case, suppose we wish to divide AB into three parts, which shall be to each other as the three lines P, Q, R.

As in the last Problem, draw any indefinite line AX, making an angle with AB. Make Ap, pq, qb respectively equal to P, Q, and R. Draw Bb, and parallel to it draw pP', qQ', and the line AB will be divided at P' and Q' as required (B. III., T. III.)



Scholium. As a particular case, suppose we wish to divide a straight line AB into two parts proportional to the lines M and N.

Through the points A and B draw any two parallel lines AX, BY, so that the alternate angles BAX, ABY may be equal. Take on AX and BY portions AC, BC', respectively equal to M and N; then draw CC' meeting AB in D.



The straight line AB will thus be divided at D in the ratio required.

For, the two triangles DAC, DBC', are evidently similar, and give

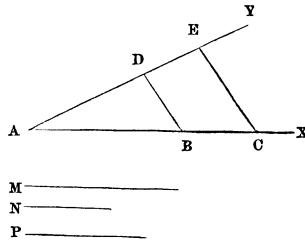
$$AD : DB :: AC : BC' :: M : N.$$

If, instead of taking BC', equal to N, in the direction of BY, opposite to AX, we had taken BC'', equal to N, in the same direction with AX, then the line CC'' would have met AB produced at D', which is called the *conjugate* point of the point D (B. III., T. XII., S.).

PROBLEM III.

To find a fourth proportional to three given lines M, N, P.

Form any angle, as XAY, and take on AX, AB = M, BC = N, and on AY, AD = P; then draw BD, and through the point C draw CE parallel to BD. The line DE will be the fourth proportional required.



For, we have (B. III., T. III.),

$$AB : BC :: AD : DE, \text{ or } M : N :: P : DE.$$

Cor. I. If P were equal to N, the above proportion would become

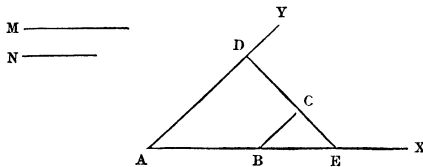
$$M : N :: N : DE.$$

That is, DE would in this case be a third proportional to the two lines M and N.

Cor. II. We deduce immediately the following problem:

A point O being given within an angle YAX, to draw through O a straight line DOE, such that the segments DO, OE may be to each other in the ratio of M to N.

Through the point O draw OB parallel to AY; find a fourth proportional to the three lines M, N, AB; make BE equal to this fourth proportional, and draw EOD, which will be the line required.



For, since OB is parallel to AD, we have

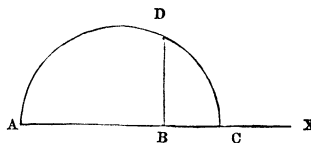
$$OD : OE :: AB : BE :: M : N.$$

When $M = N$, it is sufficient to take on AX, $BE = AB$, and then to draw EOD.

PROBLEM IV.

To find a mean proportional between two given lines M and N.

On the indefinite line AX, take $AB = M$, $BC = N$, and on AC, as a diameter, describe the semi-circumference ADC. Through B draw BD perpendicular to AC, and BD will be the mean proportional sought. For, we have (T. I., S. I.),



$$M : BD :: BD : N.$$

$$AB : BD :: BD : BC, \text{ or } M : BD :: BD : N.$$

PROBLEM V.

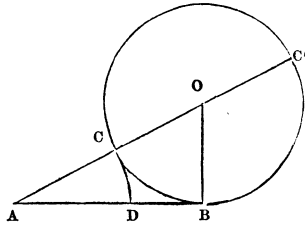
To divide a given line into mean and extreme ratio.

We have already noticed this kind of division (T. III., S. I.). And we have also actually solved this problem algebraically (T. XI., S. I.).

We will now proceed with its geometrical solution.

Let AB be the line which we are required to divide.

Draw BO perpendicular to AB and equal to one half of AB ; then draw AO , with O as a centre; with OB for a radius, describe the circumference meeting AO in C . Finally, make AD equal to AC , and the line AB will be divided at the point D as required.



In effect, from the construction, AB is a tangent to the circle OB ; and if AO is produced until it meets the circumference in C' , we shall have (T. III.),

$$AC' : AB :: AB : AC.$$

Consequently,

$$AC' - AB : AB :: AB - AC : AC.$$

Now, $AC' = AC + CC' = AC + AB$; consequently, $AC' - AB = AC = AD$, and $AB - AC = AB - AD = BD$; hence the proportion evidently becomes

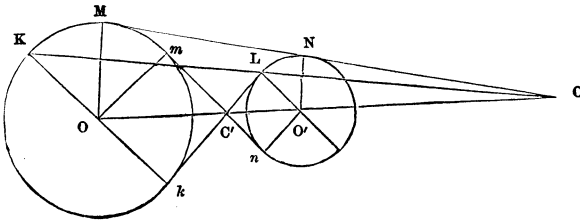
$$AD : AB :: BD : AD,$$

or interchanging means and extremes,

$$AB : AD :: AD : BD.$$

PROBLEM VI.

To draw a tangent common to two circumferences.



We will suppose the problem solved, and that MN , mn are the two tangents, the one *exterior* and the other *interior*, meeting the line of the centres OO' in two points C , C' .

It is evident that, if these points were known, it would be sufficient to draw through each of them a tangent to one of the circles (B. II., P. XII.), and it would of necessity be tangent to the other; the problem would thus become resolved.

Now, in drawing the radii OM and O'N, Om and O'n, we shall evidently obtain two couple of similar triangles OMC and ONC, OmC and O'nC', which give the proportions

$$OC : O'C :: OM : O'N,$$

$$OC' : O'C' :: OM : O'N.$$

But the radii OM, O'N, are given lines. We see then that the points C, C' are the *conjugate* points (B. III., T. XII., S.; also P. II., S.), which divide the distance OO' in the ratio of OM to O'N. From this results the following construction:

Draw any diameter KO*k* of the circle O, and through the point O' draw the radius O'L parallel to OK; join the points K and *k* with the point L.

The points C, C', where the straight lines KL, *k*L meet the line of the centres, are the points sought; since we have (B. III., T. IV.),

$$OC : O'C :: OM : O'N,$$

$$OC' : O'C' :: OM : O'N.$$

Now draw through each of these points a tangent to either of these circles, and it will be tangent to the other.

Scholium. This problem is evidently susceptible of *four, three, two, or only one* solution, or it may not admit of *any*, according to the *five* relative positions of the two circumferences (B. II., T. XVIII., S.).

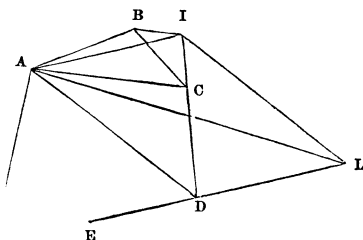
PROBLEMS OF AREAS.

PROBLEM VII.

To transform a polygon into another having one side less than the first, and finally into a triangle.

Let ABCDE, etc., be any polygon whatever, which we here represent by a broken line, in order that the generality of the construction may be the better shown.

Through the point A draw the diagonal AC cutting off the triangle ABC. Draw afterwards, through the point B, the line BI parallel to AC, meeting DC, produced, in I, then draw AI.



The polygon AIDE, etc., is equivalent to the given polygon ABCDE, etc., and it has one side less.

For, since BI is parallel to AC, the two triangles AIC, ABC are equivalent (B. III., T. XIX., C.), and if to these two triangles we add the portion of the surface ACDE, etc., we shall have, in the first case, the polygon AIDE, etc., and in the second case, the polygon ABCDE, etc.; consequently these two polygons are equivalent.

It is evident, moreover, since the side CI of the triangle AIC is the prolongation of the side DC, that the two sides AB, BC of the first polygon have been replaced by the single side AI; hence the second figure has one side less than the first.

Actually operate upon the polygon AIDE, etc., as upon the first; that is to say, draw the diagonal AD cutting off the triangle AID; afterwards draw IL parallel to AD meeting ED produced in L; we shall thus obtain a new polygon ALE, etc., equivalent to the second and having one side less.

In continuing, in this way, we shall necessarily reach a polygon of three sides, and the problem will be resolved.

PROBLEM VIII.

To transform a polygon into a square.

If the given polygon is a *triangle*, let b represent its base, and h its height, and x the side of the square sought.

We have this condition,

$$x^2 = b \times \frac{1}{2}h,$$

from which we obtain the following proportion :

$$b : x :: x : \frac{1}{2}h.$$

Thus, the *side* of the square sought is a *mean proportional* between the base and half the height of the triangle.

This line being constructed (P. IV.), we easily obtain the construction of the square.

Whatever may be the polygon, we commence by transforming it into a triangle (P. VII.) ; afterwards we transform, as above, the triangle into a square.

When the given polygon is a *parallelogram*, or a *rectangle*, or in short, any figure whose area depends immediately upon the *product of two lines*, all that is necessary in order to obtain the side of an equivalent square is *to construct a mean proportional between these two lines*.

Thus, in the case of a regular polygon, it is sufficient, after having *developed* on an indefinite straight line the perimeter of the polygon, to seek a mean proportional between the *half perimeter and the radius of its inscribed circle*.

To transform a circle into a square, it is necessary first to *rectify* the circumference, and afterwards to determine a mean proportional between the *radius* and the *rectified semi-circumference*.

Scholium. The *quadrature* of the circle is wholly dependent upon the *rectification* of a circumference ; and thus, up to the present, we have not been able, by the assistance of the Ruler and the Compass, to construct rigorously a *square equivalent to a circle*, as we can do for rectilinear figures, since all known methods give only approximate values for the *ratio of the circumference to the diameter*.

CONSTRUCTION OF SIMILAR POLYGONS UNDER CERTAIN CONDITIONS.

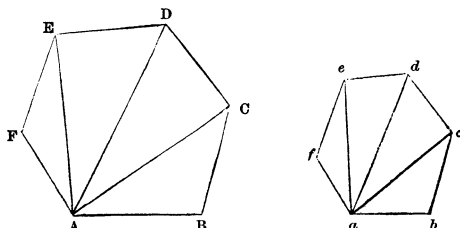
PROBLEM IX.

Upon a given line, to construct a polygon similar to a given polygon.

Let *ab* be the given line, and ABCDEF the given polygon.

After decomposing the given polygon into triangles by drawing diagonals from the corner A to the other corners, form at the

points a and b two angles cab , abc , respectively equal to the angles CAB , ABC . We shall thus obtain a triangle abc similar



to the triangle ABC . Construct, in the same manner, on ac a triangle acd similar to the triangle ACD ; and so on for the other triangles.

The polygon $abcdef$ thus obtained will be similar to the given polygon $ABCDEF$.

PROBLEM X.

Two similar polygons being given, it is required to construct a third polygon similar to the two first, and equivalent to their sum or to their difference.

The solution of this problem is an immediate consequence of C. IV., T. XXIX., B. III.

Let a , a' be two homologous sides of the given polygons A , A' . Upon these sides, regarded as the sides about the right angle, or as the hypotenuse and one of the sides of the right angle, construct a right-angled triangle; afterwards, on the third side a'' of the triangle thus obtained, construct (P. IX.) a polygon A'' similar to one of the given polygons.

The polygon thus constructed will be the polygon required.

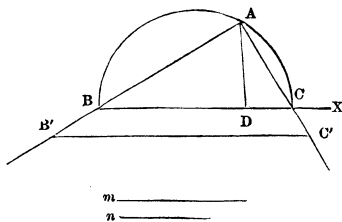
For, by construction, $a''^2 = a^2 + a'^2$, or $a''^2 = a^2 - a'^2$; hence, (B. III., T. XXIX., C. IV.),

$$A'' = A + A', \text{ or } A'' = A - A'.$$

PROBLEM XI.

To find a square which shall be to a given square in the ratio of two given lines.

On an indefinite straight line BX take $BD = m$, one of the given lines, $DC = n$, the other given line, and on $BC = m + n$, as a diameter, describe a semi-circumference; having drawn DA perpendicular to BC, draw the chords AB, AC, and prolong them if necessary beyond B and C. Upon AB take $AB' = a$, the side of the given square, which we here suppose to be greater than AB, and draw $B'C'$ parallel to BC. The line AC' will be the side of the square sought.



In effect, the two similar triangles ABC, $AB'C'$, give the proportion

$$AB : AC :: AB' : AC'.$$

Consequently, $AB^2 : AC^2 :: AB'^2 : AC'^2$;

but we have (B. III., T. XIV., S. I.),

$$AB^2 : AC^2 :: BD : DC, \text{ or } :: m : n;$$

hence,

$$AB'^2 : AC'^2 :: m : n;$$

or, putting a for AB' , and changing the order of the terms of the proportion, we have

$$m : n :: a^2 : AC'^2.$$

PROBLEM XII.

A polygon P being given, to construct a second polygon X similar to the first, and such that the first may be to the second as m to n.

Let a denote the side of the given polygon, and x the homologous side of the sought polygon. We shall have

$$P : X :: a^2 : x^2,$$

$$P : X :: m : n;$$

consequently,

$$m : n :: a^2 : x^2.$$

The question is thus made to depend upon the preceding problem; x being known, it is sufficient then to construct on this line a polygon similar to the given polygon.

PROBLEM XIII.

Having given two polygons, P and Q, to construct a third polygon X similar to the first, and such that the second may be to the third in the ratio of m to n.

Let a denote a side of the polygon P, and x the homologous side of the polygon X.

Then we shall have, by the conditions of the problem,

$$\begin{aligned} P : X &:: a^2 : x^2, \\ Q : X &:: m : n. \end{aligned}$$

This being supposed, conceive the polygons P and Q transformed into equivalent squares (P. VIII.), having respectively for their sides p and q . In the same manner let y represent the side of a square equivalent to X; the preceding proportions will be changed into the following :

$$\begin{aligned} p^2 : y^2 &:: a^2 : x^2; \\ \text{hence,} \quad p : y &:: a : x, \\ \text{and} \quad p^2 : y^2 &:: m : n. \end{aligned}$$

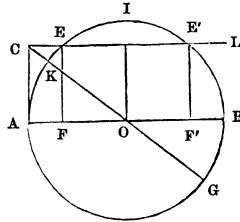
Now, the last of these proportions will make known y by the construction of Problem XI., and the second will then give x as a fourth proportional to the lines p , y , and a , which will be the side of the polygon sought, homologous to a .

PROBLEM XIV.

To construct a rectangle equivalent to a given square m^2 , and such that the sum or the difference of two adjacent sides may be equal to a given line a .

First case. The given line a being the sum of the adjacent sides.

Upon a straight line $AB = a$, as a diameter, describe a circumference; through A the extremity of the diameter AB, draw the perpendicular $AC = m$. Through C draw CL parallel to AB, meeting the circumference in the points E, E'. From the points E and E' draw EF, E'F' perpendicular to AB.



The distances AF and FB, or AF' and F'B, will be the two adjacent sides of the rectangle required to be constructed.

In effect, we have (T. I., S. I.),

$$AF \times FB = EF^2 = AC^2 = m^2,$$

and

$$AF + FB = AB = a.$$

The problem is *impossible* if $AC > OI$ or OA ; that is, when $m > \frac{1}{2}a$.

If $m = \frac{1}{2}a$, the points E, E' will unite at I, and the rectangle will become a square.

This demonstrates that *the greatest rectangle which we can form by dividing a given line into two parts for the adjacent sides, is the square constructed on half the given line.*

Second case. The given line being the difference of two adjacent sides.

After having drawn $AC = m$, as in the first case, we join C and O, and produce it so as to meet the circumference at the points K and G. The two lines CG, CK will be the adjacent sides of the rectangle sought.

We have, by the construction (T. III.),

$$CA^2 = CG \times CK, \text{ or } CG \times CK = m^2,$$

and

$$CG - CK = KG = AB = a.$$

In this case, the problem is always possible for all values of a and m .

FIFTH BOOK.

OF THE PLANE, AND ITS COMBINATION WITH THE STRAIGHT LINE.

DEFINITIONS.

I. The *intersection* of two planes is the line in which they meet to cut each other.

It is obvious, from our definition of a plane (B. I., D. XVII.), that this intersection is a straight line.

II. A line is *perpendicular to a plane*, when it is perpendicular to all the lines in that plane which meet it.

III. One *plane is perpendicular* to another, when every line in the one which is perpendicular to their intersection is perpendicular to the other plane.

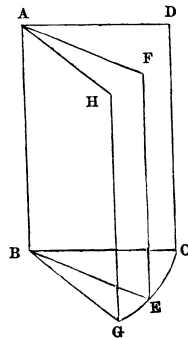
IV. A line is *parallel to a plane*, when, if both are produced to any distance, they do not meet; and, conversely, the plane is then also parallel to the line.

V. Two *planes are parallel to each other*, when, both being produced to any distance, they do not meet.

VI. The inclination of two planes which intersect, is called a *diedral angle*.

The planes forming a diedral angle are called the *faces* of the angle, and their intersection is called its *edge*.

If a rectangle ABCD is revolved about its side AB as an axis, so as to assume in succession the positions ABEF, ABGH, etc., the point C will describe the circumference of a circle. In fact, any point of the plane of this rectangle, not situated in the axis AB, will, in a similar manner, describe the circumfer-



ence of a circle having its centre in the axis. During this revolution, the arc CE will increase in the same ratio as the opening or inclination of the planes ABCD, ABEF, increases; that is, as the *diedral angle* DABE increases. Now, the arc CE is the measure of the angle CBE (B. II., T. IX., S.), hence a diedral angle has for its measure the angle contained by two lines drawn from any point of its edge, perpendicular to the same, one in each face.

VII. When several planes pass through a common point, the angular space included between these planes is called a *polyedral angle*. Each plane is called a *face*; the line in which any two faces intersect is called an *edge*; and the common point through which the planes pass is called the *vertex*.

Three planes, at least, are necessary for forming a polyedral angle. In the case of three planes, the angle is called a *triedral angle*.

THEOREM I.

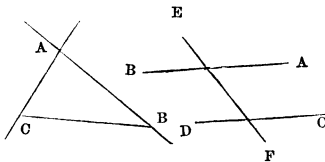
One part of a straight line cannot be in a plane, and another part out of it.

For (B. I., D. XVII.), when a straight line has two points common with a plane, it must coincide with the plane.

THEOREM II.

Two straight lines which intersect each other, are situated in the same plane, and determine its position.

Let AB, AC be two straight lines which intersect each other in A; and conceive a plane passing through one of the lines, as AB, and if also AC be in this



plane, then it is clear that the two lines, according to the terms of the proposition, are in the same plane; but if not, let the plane passing through AB be supposed to be turned round AB till it passes through the point C, then the line AC, which has two of its points, A and C, in this plane, coincides with it; and

hence the position of the plane is determined by the single condition of containing the two straight lines AB, AC.

Cor. I. A triangle ABC, or any three points not in a straight line, determines the position of a plane.

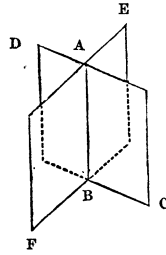
Cor. II. Hence, also, two parallels AB, CD determine the position of a plane; for, drawing the secant EF, the plane of the two straight lines AB, EF is that of the parallels AB, CD.

THEOREM III.

The intersection of two planes is a straight line.

Let DC and EF be two planes cutting each other, and A, B two points in which the planes meet. Draw the straight line AB; this line is the intersection of the two planes.

For, since the straight line touches the two planes in the points A and B, it lies wholly in both these planes, or is common to both of them; that is, the intersection of the two planes is in a straight line.



THEOREM IV.

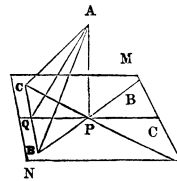
If a straight line is perpendicular to each of two straight lines at their point of intersection, it is perpendicular to the plane of these lines.

Let AP be perpendicular to the two lines PB, PC, at their point of intersection P; then will it be perpendicular to MN the plane of the lines.

Through P draw in the plane MN any line, as PQ; and through any point of this line, as Q, draw BQC, so that BQ = QC (B. IV., P. III., C. II.); join AB, AQ, AC.

The base BC being divided into two equal parts at the point Q, the triangle BPC (B. III., T. XVII.) will give

$$PC^2 + PB^2 = 2PQ^2 + 2QC^2.$$



The triangle BAC will, in like manner, give

$$AC^2 + AB^2 = 2AQ^2 + 2QC^2.$$

Taking the first equation from the second, and observing that the triangles APC, APB, which are both right-angled at P, give $AC^2 - PC^2 = AP^2$, and $AB^2 - PB^2 = AP^2$, we shall have

$$2AP^2 = 2AQ^2 - 2PQ^2,$$

or,

$$AP^2 = AQ^2 - PQ^2;$$

that is,

$$AQ^2 = AP^2 + PQ^2.$$

Hence the triangle APQ is right-angled at P, and therefore AP is perpendicular to PQ.

Scholium. Thus it is evident, not only that a straight line may be perpendicular to all the straight lines which pass through its foot in a plane, but that it always must be so, whenever it is perpendicular to two straight lines drawn in the plane.

Cor. At a given point P on a plane, it is impossible to draw more than one perpendicular to that plane. For, if there could be two perpendiculars at the same point P, draw along these two perpendiculars a plane, whose intersection with the plane MN is PQ; then those two perpendiculars would be perpendicular to the line PQ at the same point, and in the same plane, which is impossible.

It is also impossible to draw, from a given point out of a plane, two perpendiculars to that plane. For, let AP, AQ be these two perpendiculars, then the triangle APQ would have two right angles APQ, AQP, which is impossible.

THEOREM V.

If from a point without a plane, a perpendicular and several oblique lines be drawn to this plane:

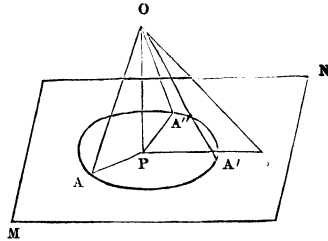
- I. *The perpendicular will be shorter than any oblique line.*
- II. *Any two oblique lines which terminate at equal distances from the foot of the perpendicular, will be equal.*
- III. *Of two oblique lines, terminating at unequal distances from the foot of the perpendicular, the one at the greater distance will be the longer.*

Let O be the given point, MN the given plane, OP perpendicular, and OA, OA', OA'', and OB oblique lines.

First. Since OA is an oblique line, the right-angled triangle OPA gives $OP < OA$ (B. I., T. XXVII.).

Secondly. If $PA = PA'$, the two right-angled triangles OPA , OPA' will be equal (B. I., T. XX., C.), and give $OA' = OA$.

Thirdly. If $PB > PA$, take on PB a distance $PA' = PA$, and draw OA' , and we shall have $OB > OA'$ (B. I., T. XII.); but we have $OA' = OA$, consequently $OB > OA$.



Cor. I. The perpendicular drawn from any point to a plane, measures the true distance of the point from the plane.

Cor. II. Any point whatever in a line perpendicular to a plane, is equally distant from all the points of the circumference described about the foot of the perpendicular as a centre, with any radius whatever.

Scholium I. This perpendicular is called the *axis* of the circle. All the points of the axis may be regarded as centres of the circle, and the oblique lines as corresponding radii, any one of which may be used for describing the circle.

Scholium II. By means of an extended thread, one extremity being fixed at the point O , and the other extremity attached to a pencil, we may determine any three points, as A , A' , A'' , equally distant from the point O . If we then describe the circumference of a circle passing through these three points, its centre will be the foot of the perpendicular drawn from O to the plane.

Scholium III. When a plane passes perpendicularly through the middle of a straight line :

1. All points of this plane are equally distant from the extremities of this line.

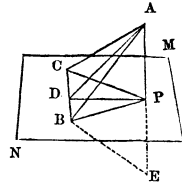
2. All points situated without this plane are unequally distant from these extremities.

THEOREM VI.

If, from a point without a plane, a perpendicular be drawn to that plane, and from the foot of the perpendicular a line be drawn perpendicular to a line in the plane, and from the point of intersection a line be drawn to the first point, this last line will be perpendicular to the line in the plane.

Let AP be a line perpendicular to the plane MN , and PD perpendicular to BC ; then will AD be perpendicular to BC .

Take $DB = DC$, and join PB, PC, AB, AC . Since PD is perpendicular at D , the middle point of BC , PB is equal to PC (B. I., T. XII.); consequently the oblique lines AB, AC are equal, and the two triangles ADB, ADC have the three sides of the one equal to the three sides of the other; therefore they are equal (B. I., T. XXV.), and the angle $ADB = ADC$; hence each is a right angle, and AD is perpendicular to BC (B. I., D. XII.).



Cor. It is evident, likewise, that BC is perpendicular to the plane APD , since BC is at once perpendicular to the two straight lines AD, PD (T. IV.).

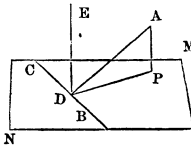
Scholium. The two straight lines AE, BC afford an instance of two lines not parallel which do not meet, because they are not situated in the same plane. The shortest distance between these lines is the straight line PD , which is perpendicular both to the line AP and to the line BC . For if we join any other two points, such as A and B , we shall have $AB > AD, AD > PD$, therefore $AB > PD$.

The two lines AE, CB , though not situated in the same plane, are conceived as forming a right angle with each other, because AE and the line drawn through one of its points parallel to BC would make with each other a right angle. In the same manner, the line AB and the line PD , which represent any two straight lines not situated in the same plane, are supposed to form with each other the same angle which would be formed by AB and a straight line parallel to PD drawn through any point of AB .

THEOREM VII.

If one of two parallel lines is perpendicular to a plane, the other will also be perpendicular to this plane.

Let AP and ED be parallel lines, of which AP is perpendicular to the plane MN ; then will ED be also perpendicular to this plane.



Along the parallels AP , ED extend a plane; its intersection with the plane MN will be PD . In the plane MN draw BC perpendicular to PD , and join AD .

Then BC is perpendicular to the plane $APDE$ (T. VI., Cor.), and therefore the angle BDE is right. But the angle EDP is right also, since AP is perpendicular to PD , and DE parallel to AP (B. I., T. XV., Cor.); therefore the line DE is perpendicular to the two straight lines DP , DB , hence it is perpendicular to their plane MN (T. IV.).

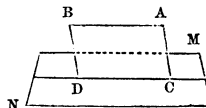
Cor. I. Conversely, if the straight lines AP , DE are perpendicular to the same plane MN , they will be parallel. For, if they be not so, draw through the point D a line parallel to AP ; this parallel will be perpendicular to the plane MN ; therefore, through the same point D more than one perpendicular might be drawn to the same plane, which (T. IV., C.) is impossible.

Cor. II. Two lines, A and B , parallel to a third line C , are parallel to each other. For, conceive a plane perpendicular to the line C ; the lines A and B , being parallel to C , will be perpendicular to the same plane; therefore, by the preceding Corollary, they will be parallel to each other.

THEOREM VIII.

If a straight line without a plane is parallel to a line in the plane, it is parallel to the plane itself.

Let the straight line AB , without the plane MN , be parallel to the line CD of this plane; then will AB be parallel to the plane MN .



For, if the line AB , which lies in the plane $ABDC$, could

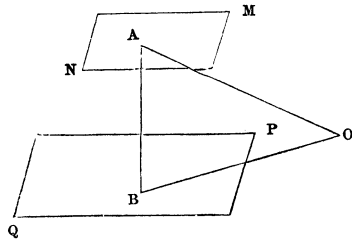
meet the plane MN , this could only be in some point of the line CD , the intersection of the two planes; but AB cannot meet CD , since they are parallel; hence it will not meet the plane MN , therefore (D. IV.) it is parallel to that plane.

THEOREM IX.

If two planes are perpendicular to the same line, they are parallel.

Let the planes MN and PQ be each perpendicular to AB ; then will they be parallel.

For, if they can meet anywhere, let O be one of their common points, and join OA , OB . The line AB , which is perpendicular to the plane MN , is perpendicular to the straight line AO drawn through its foot in that plane. For the same reason, AB is perpendicular to BO . Therefore OA and OB are two perpendiculars drawn from the same point O , upon the same straight line, which is impossible; hence the planes MN , PQ cannot meet each other, and consequently they are parallel.

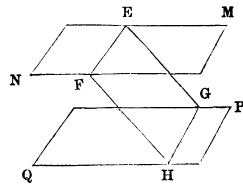


THEOREM X.

The intersections of two parallel planes with a third plane are parallel.

Let the two parallel planes MN and PQ intersect the plane EH ; then will EF be parallel to HG .

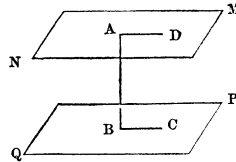
For, if the lines EF , HG , lying in the same plane, were not parallel, they would meet each other when produced; therefore the planes MN , PQ , in which those lines are situated, would also meet, and the planes would not be parallel.



THEOREM XI.

A line which is perpendicular to one of two parallel planes, is perpendicular to the other also.

Let the two planes MN and PQ be parallel; then if the line AB is perpendicular to the plane MN, it will also be perpendicular to PQ.

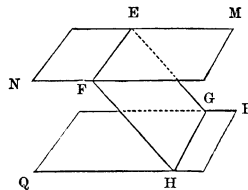


Having drawn through B any line BC in the plane PQ, along the lines AB and BC, extend a plane ABC, intersecting the plane MN in AD; the intersection AD will be parallel to BC (T. X.). But the line AB, being perpendicular to the plane MN, is perpendicular to the straight line AD; therefore, also, it is perpendicular to its parallel BC (B. I., T. XV., Cor.). Hence the line AB, being perpendicular to any line BC drawn through its foot in the plane PQ, is consequently perpendicular to that plane.

THEOREM XII.

Two parallel lines, included between two parallel planes, are equal.

Let EG, FH be two parallel lines included between the two parallel planes MN, PQ; then will these lines be equal.



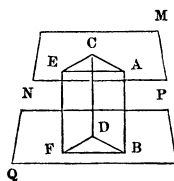
Through the parallels EG, FH, draw the plane EGFH to meet the parallel planes in EF and GH. The intersections EF, GH (T. X.) are parallel to each other, so likewise are EG, FH; therefore the figure EGFH is a parallelogram, and $EG = FH$.

Cor. Hence it follows, that *two parallel planes are everywhere equidistant*; for, if EG and FH are perpendicular to the two planes MN, PQ, they will be parallel to each other (T. VII., C.), and therefore equal.

THEOREM XIII.

If two angles, not situated in the same plane, have their sides parallel and lying in the same direction, they will be equal, and the planes in which they are situated will be parallel.

Let $\angle CAE$, $\angle DBF$ be two angles not situated in the same plane, having AC parallel to BD and lying in the same direction, and AE parallel to BF , and also lying in the same direction; then will these angles be equal, and their planes will be parallel.



Make $AC = BD$, $AE = BF$; and join CE , DF , AB , CD , EF . Since AC is equal and parallel to BD , the figure $ABDC$ is a parallelogram (B. I., T. XXX.), therefore CD is equal and parallel to AB . For a similar reason, EF is equal and parallel to AB ; hence, also, CD is equal and parallel to EF . The figure $CEFD$ is therefore a parallelogram, and the side CE is equal and parallel to DF ; therefore the triangles $\angle CAE$, $\angle DBF$ have their corresponding sides equal, and consequently the angle $\angle CAE = \angle DBF$.

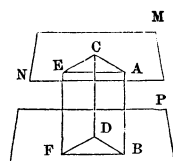
Again, the plane ACE is parallel to the plane BDF . For, suppose the plane parallel to BDF , drawn through the point A , were to meet the lines CD , EF in points different from C and E , for instance in G and H , then (T. XII.) the three lines AB , GD , HF would be equal. But the lines AB , CD , EF are already known to be equal, hence $CD = GD$, and $HF = EF$, which is absurd; hence the plane ACE is parallel to BDF .

Cor. If two parallel planes MN , PQ are met by two other planes not parallel $CABD$, $EABF$, the angles $\angle CAE$, $\angle DBF$ formed by the intersections of the parallel planes will be equal; for (T. X.) the intersection AC is parallel to BD , and AE to BF , and therefore the angle $\angle CAE = \angle DBF$.

THEOREM XIV.

If three straight lines, not situated in the same plane, are equal and parallel, the triangles formed by joining their corresponding opposite extremities will be equal, and their planes will be parallel.

Let ACE, BDF be two triangles formed by joining the opposite extremities of the three equal and parallel lines AB, CD, EF; then will these triangles be equal, and their planes will be parallel.



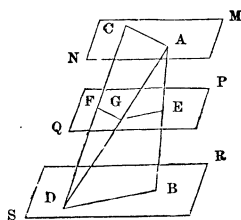
For, since AB is equal and parallel to CD, the figure ABDC is a parallelogram (B. I., T. XXX.); hence the side AC is equal and parallel to BD. For a like reason, the sides AE, BF are equal and parallel, as also CE, DF. Therefore the two triangles ACE, BDF are equal; and, consequently, as in the last proposition, their planes are parallel.

THEOREM XV.

If two straight lines are cut by three parallel planes, they will be divided proportionally.

Suppose the line AB to be cut by the parallel planes MN, PQ, RS, at the points A, E, B; and the line CD to be cut by the same planes at the points C, F, D; then

$$AE : EB :: CF : FD.$$



Draw AD meeting the plane PQ in G, and join AC, EG, GF, BD. The intersections EG, BD, of the parallel planes PQ, RS in the plane ABD, are parallel (T. X.); therefore,

$$AE : EB :: AG : GD \quad (\text{B. III., T. III.}).$$

In like manner, the intersections AC, GF being parallel,

$$AG : GD :: CF : FD.$$

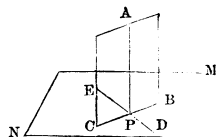
The ratio AG : GD is the same in both; hence

$$AE : EB :: CF : FD.$$

THEOREM XVI.

If a straight line is perpendicular to a plane, then every plane passing through this line will also be perpendicular to the first plane.

Let the line AP be perpendicular to the plane MN ; then any plane, as AB , passing through this line, will also be perpendicular to the plane MN .



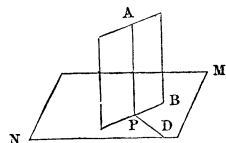
For, let BC be the intersection of the planes AB , MN . In the plane MN draw DE perpendicular to BC ; then the line AP , being perpendicular to the plane MN , will be perpendicular to each of the two straight lines BC , DE . But the angle APD , formed by the two perpendiculars PA , PD at their common intersection P , is the measure of the angle of the two planes (D. VI.); and since, in the present case, the angle is a right angle, the two planes are perpendicular to each other.

Scholium. When the three lines, such as AP , BP , DP , are perpendicular to each other, each of these lines is perpendicular to the plane of the other two, and the planes themselves are perpendicular to each other.

THEOREM XVII.

If two planes are perpendicular to each other, every line drawn in one of them perpendicular to their common intersection, will be perpendicular to the other plane.

Let the planes AB , MN be perpendicular to each other; and in the plane AB , let PA be drawn perpendicular to the common intersection PB ; then will it be perpendicular to the plane MN .



For, in the plane MN draw PD perpendicular to PB ; then because the planes are perpendicular, the angle APD is a right angle; therefore the line AP is perpendicular to the two straight lines PB , PD , and consequently perpendicular to their plane MN .

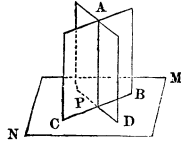
Cor. If the plane AB be perpendicular to the plane MN , and if at a point P of the common intersection a perpendicular be drawn to the plane MN , that perpendicular will be in the plane AB . For, if not, then in the plane AB we might draw AP perpendicular to PB their common intersection, and this AP at the same time would be perpendicular to the plane MN ; therefore, at the same point P there would be two perpendiculars to the plane MN , which is impossible.

THEOREM XVIII.

If two planes be perpendicular to a third plane, their common intersection will be also perpendicular to the third plane.

Let AB , AD be perpendicular to MN ; then will their common intersection AP be perpendicular to the same plane MN .

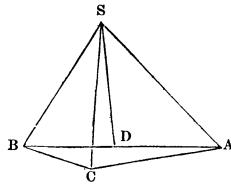
For, at the point P draw the perpendicular to the plane MN ; then that perpendicular must be in the plane AD ; it must also be in the plane AB (T. XVII.), therefore it is their common intersection AP .



THEOREM XIX.

If a polyedral angle is formed by three plane angles, the sum of any two of these angles will be greater than the third.

The proposition requires demonstration only when the plane angle, which is compared to the sum of the other two, is greater than either of them. Therefore, suppose the polyedral angle S to be formed by three plane angles ASB , ASC , BSC , whereof the angle ASB is the greatest: we are to show that $ASB < ASC + BSC$.



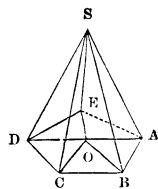
In the plane ASB make the angle $BSD = BSC$; draw the straight line ADB at pleasure; and, having taken $SC = SD$, join AC , BC .

The two sides BS , SD are equal to the two sides BS , SC ; the angle $BSD = BSC$; therefore the triangles BSD , BSC are equal, and $BD = BC$. But $AB < AC + BC$; taking BD from the one side, and from the other its equal BC , there remains $AD < AC$. The two sides AS , SD are equal respectively to the two AS , SC ; the third side AD is less than the third side AC ; therefore the angle $ASD < ASC$. Adding $BSD = BSC$, we shall have $ASD + BSD$, or $ASB < ASC + BSC$.

THEOREM XX.

The sum of the plane angles which form any polyedral angle, is always less than four right angles.

Conceive the polyedral angle S to be cut by any plane $ABCDE$; from O a point in that plane, draw to the several angles straight lines AO, OB, OC, OD, OE .



The sum of the angles of the triangles ASB, BSC , etc., formed about the vertex S , is equivalent to the sum of the angles of an equal number of triangles AOB, BOC , etc., formed about the point O . But at the point B the angles ABO, OBC , taken together, make the angle ABC (T. XIX.) less than the sum of the angles ABS, SBC . In the same manner, at the point C we have $BCO + OCD < BCS + SCD$, and so with all the angles of the polygon $ABCDE$. Whence it follows that the sum of all the angles at the bases of the triangles whose common vertex is in O , is less than the sum of all the angles at the bases of the triangles whose common vertex is in S ; hence, to make up the deficiency, the sum of the angles formed about the point O , is greater than the sum of the angles about the point S . But the sum of the angles about the point O is equal to four right angles (B. I., T. I., C. III.); therefore the sum of the plane angles, which form the polyedral angle S , is less than four right angles.

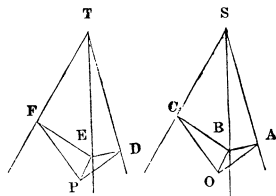
Scholium. This demonstration is founded on the supposition that the polyedral angle is convex, or that the plane of no one surface produced can ever meet the polyedral angle. If it were otherwise, the sum of the plane angles would no longer be limited, and might be of any magnitude.

THEOREM XXI.

If two polyedral angles are composed of three plane angles respectively equal to each other, the planes which contain the equal angles will be equally inclined to each other.

Let the angle $ASC = DTF$, the angle $ASB = DTE$, and the angle $BSC = ETF$; then will the inclination of the planes ASC, ASB be equal to that of the planes DTF, DTE .

Having taken SB at pleasure, draw BO perpendicular to the plane ASC ; from the point O at which



that perpendicular meets the plane, draw OA , OC perpendicular to SA , SC ; join AB , BC ; next take $TE = SB$; draw EP perpendicular to the plane DTF ; from the point P draw PD , PF perpendicular to TD , TF ; lastly, join DE , EF .

The triangle SAB is right-angled at A , and the triangle TDE is right-angled at D (T. VI.), and since the angle $ASB = DTE$, we have $SBA = TED$. Likewise $SB = TE$, therefore the triangle SAB is equal to the triangle TDE ; therefore $SA = TD$, and $AB = DE$. In like manner it may be shown that $SC = TF$, and $BC = EF$. That granted, the quadrilateral $SAOC$ is equal to the quadrilateral $TDPF$; for, place the angle ASC upon its equal DTF , because $SA = TD$, and $SC = TF$, the point A will coincide with D , and the point C with F ; and at the same time AO , which is perpendicular to SA , will coincide with PD , which is perpendicular to TD , and in like manner CO with FP ; wherefore the point O will coincide with the point P , and AO will be equal to DP . But the triangles AOB , DPE are right-angled at O and P ; the hypotenuse $AB = DE$, and the side $AO = DP$; hence those triangles are equal—therefore the angle $OAB = PDE$. The angle OAB is the inclination of the two planes ASB , ASC , and the angle PDE is that of the two planes DTE , DTF ; hence these two inclinations are equal to each other.

It must, however, be observed, that the angle A of the right-angled triangle AOB is properly the inclination of the two planes ASB , ASC , only when the perpendicular BO falls on the same side of SA as SC falls; for if it fell on the other side, the angle of the two planes would be obtuse, and, added to the angle A of the triangle OAB , it would make two right angles. But, in the same case, the angle of the two planes TDE , TDF would also be obtuse, and, added to the angle D of the triangle PDE , it would make two right angles; and the angle A being thus always equal to the angle at D , it would follow, in the same manner, that the inclination of the two planes ASB , ASC must be equal to that of the two planes DTE , DTF .

SIXTH BOOK.

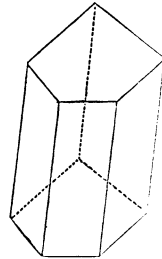
BODIES BOUNDED BY PLANES.

DEFINITIONS.

I. A solid, bounded by polygons, is called a *Polyedron*. The bounding polygons are called *faces*, the straight lines formed by the intersection of any two faces are called *edges*, and the points where three or more faces meet are called *corners*.

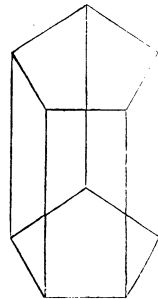
A straight line joining any two corners not situated in the same face is called a *diagonal*.

II. A polyedron, having two of its faces equal and parallel, and all the other faces parallelograms, is called a *Prism*. The two equal and parallel faces are called the *bases* of the prism, usually referred to as the *lower base* and the *upper base*; and the parallelograms which make up the other faces, taken together, constitute the *lateral surface* of the prism.

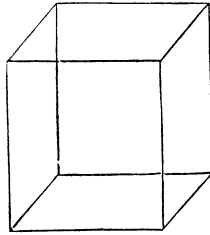


III. A prism is *triangular*, *quadrangular*, *pentagonal*, etc., according as the bases are triangles, quadrilaterals, pentagons, etc.

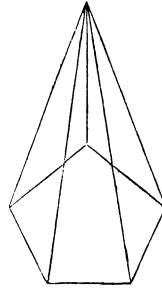
IV. When the edges formed by the intersection of the lateral faces of a prism are perpendicular to its bases, it is a *right prism*: each edge is then equal to its altitude. In all other cases the prism is said to be *oblique*, and the edges are greater than the altitude.



V. A prism, whose bases are parallelograms, is called a *Parallelopipedon*. When all the faces are rectangles it is called a *Rectangular Parallelopipedon*. When all the faces are squares it is called a *Cube*, or *regular hexaedron*.



VI. A *Pyramid* is a solid formed by several triangular planes meeting in a point, called the *vertex*, and terminating in the sides of a polygon, forming its *base*. The triangles meeting at the vertex, taken together, constitute the *lateral surface* of the pyramid. The perpendicular distance from the vertex to the base is called its *altitude*.

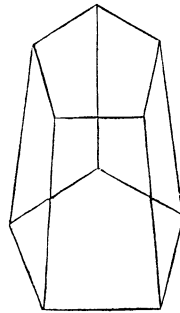


VII. A pyramid is *triangular*, *quadrilateral*, *pentagonal*, etc., according as the base is a triangle, quadrilateral, pentagon, etc.

VIII. When the base of a pyramid is a regular polygon, and the triangles forming the lateral surface are isosceles, it is a *right pyramid*. The perpendicular drawn from the vertex to the base will, in this case, pass through its centre, and it is then called the *axis* of the pyramid. A line drawn from the vertex of a right pyramid to the middle point of one of the sides of the polygon constituting its base, is called its *slant height*.

IX. A triangular pyramid, having in all four triangular faces, is called a *Tetraedron*.

X. If a pyramid is cut by a plane parallel to its base it will be divided into two portions, one of which will be a second pyramid having the same vertex as the first. The remaining portion, having two parallel *bases*, is called a *truncated pyramid*, or *frustum of a pyramid*. The *altitude* of a frustum of a pyramid is the perpendicular distance between its bases. The *slant height* of a frustum of a right pyramid is the portion of its slant height included between its bases.



XI. *Two tetraedrons are similar* when they have their edges proportional and arranged in the same order. Consequently the triangular faces of two similar tetraedrons are similar triangles each to each (B. III., D. II.).

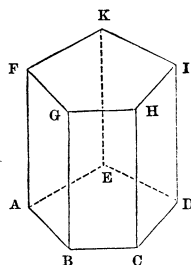
Any two polyedrons whatever, are similar when they are capable of being decomposed into the same number of similar tetraedrons each to each, having the same order of arrangement.

XII. A *regular Polyedron* is one having all its faces equal and regular polygons; and all its diedral angles equal.

THEOREM I.

The lateral surface of a right prism is equal to the perimeter of its base multiplied by its altitude.

For this surface is equal to the sum of the rectangles AFGB, BGHC, CHID, etc., which compose it. Now the altitudes AF, BG, CH, etc., of those rectangles, are equal to the altitude of the prism; their bases AB, BC, CD, etc., taken together, make up the perimeter of the prism's base: hence the sum of these rectangles, or the lateral surface of the prism, is equal to the perimeter of its base multiplied by its altitude.



Cor. If two right prisms have the same altitude, their lateral surfaces will be to each other as the perimeters of their bases.

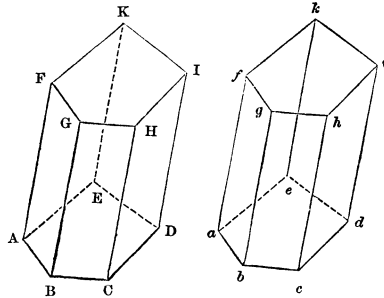
THEOREM II.

Two prisms are equal when a polyedral angle in each is contained by three planes, which are respectively equal in both, and similarly situated.

Let the base ABCDE be equal to the base *abcde*; the parallelogram ABGF equal to the parallelogram *abgf*, and the parallelogram BCHG equal to the parallelogram *bchg*; then will the prism ABCI be equal to the prism *abci*.

For, apply the base ABCDE upon its equal *abcde*, so that

they may coincide. But the three plane angles which form the polyedral angle B are respectively equal to the three plane angles which form the polyedral angle $\hat{b}$; that is, $\angle ABC = \angle abc$, $\angle ABG = \angle abg$, and $\angle GBC = \angle gbc$, and they are also similarly situated; therefore the poly-



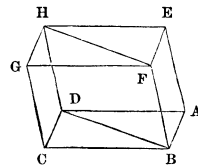
edral angles at B and $\hat{b}$ are equal (B. V., T. XXI.); hence BG will coincide with its equal bg . And it is likewise evident, since the parallelograms ABGF and $abgf$ are equal, that the side GF will coincide with its equal gf : and in the same manner GH will coincide with gh ; therefore the upper base FGHK will coincide with its equal $fghik$, and the two solids will be identical, since their vertices are the same.

Cor. Two right prisms, which have equal bases and equal altitudes, are equal. For, since the side AB is equal to ab , and the altitude BG to bg , the rectangle ABGF will be equal to $abgf$, and in the same way, the rectangle BGHC will be equal to $bghe$; and thus the three planes which form the solid angle B will be equal to the three planes which form the solid angle $\hat{b}$: hence the two prisms are equal.

THEOREM III.

In every parallelepipedon the opposite faces are equal and parallel.

By the definition of this solid, the bases ABCD, EFGH are equal parallelograms, and their sides are parallel: it remains only to show that the same is true of any two opposite lateral faces, such as AEHD, BFGC. Now AD is equal and parallel to BC, because the figure ABCD is a parallelogram; for a like reason, AE is parallel to BF: hence the angle DAE is equal to the angle CBF, and the planes DAE, CBF are parallel; hence also the parallelogram DAEH is equal to the parallelogram CBFH.



In the same way, it may be shown that the opposite parallelograms ABFE, DCGH are equal and parallel.

Cor. Since the parallelepipedon is a solid bounded by six planes, whereof those lying opposite to each other are equal and parallel, it follows that any face and the one opposite to it may be assumed as the bases of the parallelepipedon.

Scholium. If three straight lines AB, AE, AD, passing through the same point A, and making given angles with each other, are known, a parallelepipedon may be formed on those lines. For this purpose a plane must be extended through the extremity of each line, and parallel to the plane of the other two; that is, through the point B a plane parallel to DAE, through D a plane parallel to BAE, and through E a plane parallel to BAD. The mutual intersections of these planes will form the parallelepipedon required.

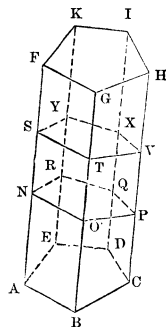
THEOREM IV.

In every prism the sections formed by parallel planes are equal polygons.

In the prism ABCI, let the sections NOPQR, STVXY be formed by parallel planes; then will these sections be equal polygons.

For the sides ST, NO are parallel, being the intersections of two parallel planes with a third plane ABGF; moreover the sides ST, NO are included between the parallels NS, OT, which are sides of the prism: hence NO is equal to ST. For like reasons, the sides OP, PQ, QR, etc., of the section NOPQR, are respectively equal to the sides TV, VX, XY, etc., of the section STVXY; and since the equal sides are at the same time parallel, it follows that the angles NOP, OPQ, etc., of the first section are respectively equal to the angles STV, TVX, etc., of the second: hence the two sections NOPQR, STVXY are equal polygons.

Cor. Every section in a prism, if made parallel to the base, is also equal to that base.



THEOREM V.

If a plane be made to pass through the diagonal and opposite edges of a parallelopipedon, so as to divide it into two triangular prisms, those prisms will be equal.

Let the parallelopipedon $ABCG$ be divided by the plane $BDHF$ into the two triangular prisms $ABDHEF$, $BCDFGH$; then will those prisms be equal.

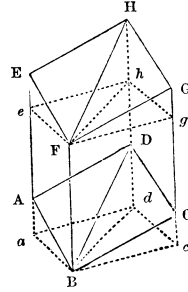
Through the vertices B and F , draw the planes $Badc$, $Fehg$ at right angles to the side BF , and meeting AE , DH , CG , the three other sides of the parallelopipedon, in the points a , d , c towards one direction, and in e , h , g towards the other: then the sections $Badc$, $Fehg$ will be equal parallelograms; being equal, because they are formed by planes perpendicular to the same straight line, and consequently parallel; and being parallelograms, because aB , dc , two opposite sides of the same section, are formed by the meeting of one plane with two parallel planes $ABFE$, $DCGH$.

For a like reason, the figure $BaeF$ is a parallelogram; so also are $BFgc$, $cdhg$, and $adhe$, the other lateral faces of the solid $BadcFehg$: hence that solid is a prism (D. IV.), and that prism is a right one, because the side BF is perpendicular to its base.

This being proved, if the right prism Bh be divided by the plane $BFHD$ into two right-triangular prisms, $aBdeFh$, $BdcFhg$, it will remain to be shown that the oblique-triangular prism $ABDEFH$ will be equal to the right-triangular prism $aBdeFh$; and since those two prisms have a part $ABDheF$ in common, it will only be requisite to prove that the remaining parts, namely, the solids $BaADD$, $FeEHh$, are equal.

Now, by reason of the parallelograms $ABFE$, $aBFe$, the sides AE , ae being equal to their parallel BF , are equal to each other; and taking away the common part Ae , there remains $Aa = Ee$. In the same manner we could prove $Dd = Hh$.

Let us now place the base Feh on its equal Bad ; the point e coinciding with a , and the point h with d , the sides eE , hH will coincide with their equals aA , dD , because they are perpendicular to the same plane Bad . Hence the two solids in ques-



tion will coincide exactly with each other, and the oblique prism $BADFEH$ is therefore equal to the right one $BadFeh$.

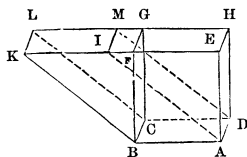
In the same manner might the oblique prism $BDCFHG$ be proved equal to the right prism $BdcFhg$. But (T. I.) the two right prisms $BadFeh$, $BdcFhg$ are equal, since they have the same altitude BF , and since their bases Bad , Bdc are halves of the same parallelogram. Hence the two triangular prisms $BADFEH$, $BDCFHG$, being equal to the equal right prisms, are equal to each other.

Cor. Every triangular prism $ABDHEF$ is half of the parallelopipedon AG described with the same polyedral angle A , and with the same edges AB , AD , AE .

THEOREM VI.

Two parallelopipedons having a common base, and their upper bases in the same plane and between the same parallels, are equivalent to each other.

Let the two parallelopipedons AG , AL have the common base $ABCD$; and let their upper bases $EFGH$, $IKLM$ be in the same plane, and between the same parallels EK , HL ; then will these parallelopipedons be equivalent.



There may be three cases to this proposition, according as EI is greater, less than, or equal to EF ; but the demonstration is the same for all. In the first place, then, we shall show that the triangular prism $AEIDHM$ is equal to the triangular prism $BFKCGL$.

Since AE is parallel to BF , and HE to GF , the angle $AEI = BFK$, $HEI = GFK$, and $HEA = GFB$. Of these six angles, the first three form the polyedral angle E , and the last three the polyedral angle F ; therefore, the plane angles being respectively equal and similarly arranged, the polyedral angles F and E must be equal. Now if the prism AEM be laid on the prism BFL , the base AEI being placed on the base BFK , will coincide with it, because they are equal; and since the polyedral angle E is equal to the polyedral angle F , the side EH will coincide with its equal FG ; and nothing more is required to prove the coinci-

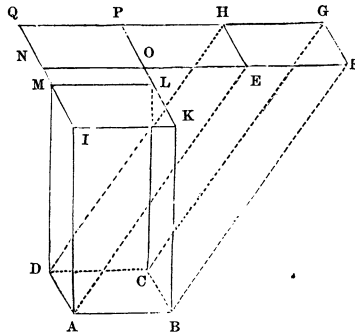
dence of the two prisms throughout their whole extent, for (T. II.) the base AEI and the edge EH determine the prism AEM, as the base BFK and the edge FG determine the prism BFL; hence these prisms are equal.

But if the prism AEM is taken away from the solid AL, there will remain the parallelepipedon AIL; and if the prism BFL is taken away from the same solid, there will remain the parallelepipedon AEG; hence these two parallelepipedons AIL, AEG are equivalent.

THEOREM VII.

Two parallelepipedons having the same base and same altitude, are equivalent to each other.

Let ABCD be the common base of the two parallelepipedons AG, AL: since they have the same altitude, their upper bases EFGH, IKLM will be in the same plane. Also the sides EF and AB will be equal and parallel, as well as IK and AB; hence EF is equal and parallel to IK: for a like reason, GF is equal and parallel to LK. Let

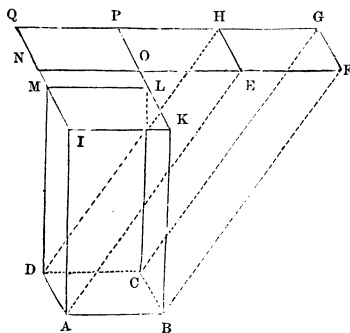


the sides EF, HG be produced, and likewise LK, IM, till, by their intersections, they form the parallelogram NOPQ: this parallelogram will evidently be equal to either of the bases EFGH, IKLM. Now, if a third parallelepipedon be conceived, having ABCD for its lower base and NOPQ for its upper, this third parallelepipedon will (T. VI.) be equivalent to the parallelepipedon AG; since, with the same lower base, their upper bases lie in the same plane and between the same parallels GQ, FN. For the same reason, this third parallelepipedon will also be equivalent to the parallelepipedon AL; hence the two parallelepipedons AG, AL, which have the same base and the same altitude, are equivalent.

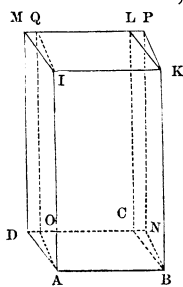
THEOREM VIII.

Any parallelopipedon may be changed into an equal rectangular parallelopipedon having the same altitude and an equivalent base.

Let AG be the parallelopipedon proposed. From the points A, B, C, D , draw AI, BK, CL, DM perpendicular to the plane of the base; and we shall thus form the parallelopipedon AL equivalent to AG , and having its lateral faces AK, BL , etc., rectangular. Hence, if the base $ABCD$ be a rectangle, AL will be the rectangular parallelopipedon equivalent to AG , the parallelopipedon proposed.



But if $ABCD$ is not a rectangle, draw AO and BN perpendicular to CD , and OQ and NP perpendicular to the base; then the solid $ABNOIKPQ$ will be a rectangular parallelopipedon; for, by construction, the base $ABNO$ and its opposite $IKPQ$ are rectangles; so also are the lateral faces, the edges AI, OQ , etc., being perpendicular to the plane of the base; hence the solid AP is a rectangular parallelopipedon. But the two parallelopipedons AP, AL may be conceived as having the same base $ABKI$, and the same altitude AO ; hence the parallelopipedon AG , which was at first changed into an equivalent parallelopipedon AL , is again changed into an equivalent rectangular parallelopipedon AP , having the same altitude AI , and a base $ABNO$ equivalent to the base $ABCD$.

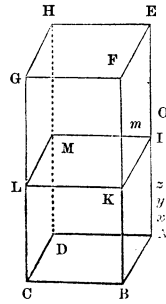


THEOREM IX.

Two rectangular parallelopipeds having the same base, are to each other as their altitudes.

Let the parallelopipeds AG, AL have the common base ABCD; then will they be to each other as their altitudes AE, AI.

First, suppose the altitudes AE, AI to be to each other as two whole numbers; for example, as 15 is to 8. Divide AE into 15 equal parts, whereof AI will contain 8; and through x, y, z , etc., the points of division, draw planes parallel to the base. These planes will cut the solid AG into 15 partial parallelopipeds, all equal to each other, having equal bases and equal altitudes: equal bases, because every section MIKL made parallel to the base ABCD of a prism, is equal to that base (T. IV., C.); and equal altitudes, because these altitudes are the equal divisions Ax, xy, yz , etc. But of those 15 equal parallelopipeds, 8 are contained in AL; hence the solid AG is to the solid AL as 15 is to 8; or, generally, as the altitude AE is to the altitude AI.



Again, if the ratio of AE to AI cannot be expressed in whole numbers, it is to be shown that, notwithstanding, we shall have
solid AG : solid AL :: AE : AI.

For, if this proportion is not correct, suppose we have

$$\text{solid AG : solid AL :: AE : AO,}$$

greater than AI. Divide AE into equal parts, such that each shall be less than OI; there will be at least one point of division m between O and I. Let P be the parallelopipedon, whose base is ABCD and altitude Am . Since the altitudes AE, Am are to each other as two whole numbers, we shall have

$$\text{solid AG : P :: AE : } Am.$$

But, by hypothesis, we have

$$\text{solid AG : solid AL :: AE : AO;}$$

therefore, solid AL : P :: AO : Am .

But AO is greater than Am : hence, if the proportion is correct, the solid AL must be greater than P. On the contrary, how-

ever, it is less; hence the fourth term of this proportion, solid $AG : \text{solid } AL :: AE : x$, cannot possibly be a line greater than AI .

By the same mode of reasoning, it might be shown that the fourth term cannot be less than AI ; therefore it is equal to AI . Hence rectangular parallelopipeds having the same base, are to each other as their altitudes.

THEOREM X.

Two rectangular parallelopipeds having the same altitude, are to each other as their bases.

Let the two rectangular parallelopipeds AG, AK have the same altitude; then will they be to each other as their bases $ABCD, AMNO$.

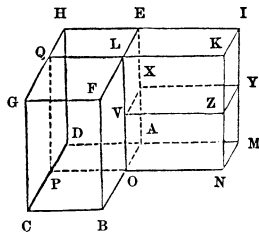
Produce the plane $ONKL$ till it meets the plane $DCGH$ in PQ ; we shall thus have a third parallelopipedon AQ , which may be compared with each of the parallelopipeds AG, AK . The two solids AG, AQ having the same base $AEHD$, are to each other as their altitudes AB, AO . In like manner, the two solids AQ, AK having the same base $AOLE$, are to each other as their altitudes AD, AM . Hence we have the two proportions

$$\begin{aligned} \text{sol. } AG : \text{sol. } AQ &:: AB : AO; \\ \text{sol. } AQ : \text{sol. } AK &:: AD : AM. \end{aligned}$$

Multiply together the corresponding terms of these proportions, omitting in the result the common multiplier $\text{sol. } AQ$: we shall have

$$\text{sol. } AG : \text{sol. } AK :: AB \times AD : AO \times AM.$$

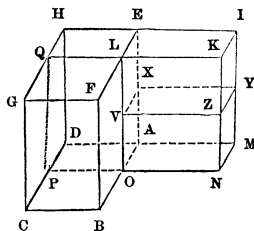
But $AB \times AD$ represents the base $ABCD$, and $AO \times AM$ represents the base $AMNO$; hence two rectangular parallelopipeds of the same altitude are to each other as their bases.



THEOREM XI.

Any two rectangular parallelopipedons are to each other as the products of their bases by their altitudes; that is to say, as the products of their three dimensions.

For, having placed the two solids AG, AZ, so that their surfaces have the common angle BAE, produce the interior planes necessary for completing the third parallelopipedon AK, having the same altitude with the parallelopipedon AG. By the last proposition, we shall have



$$\text{sol. AG : sol. AK :: ABCD : AMNO.}$$

But the two parallelopipedons AK, AZ having the same base AMNO, are to each other as their altitudes AE, AX; hence we have

$$\text{sol. AK : sol. AZ :: AE : AX.}$$

Multiply together the corresponding terms of these proportions, omitting in the result the common multiplier sol. AK: we shall have

$$\text{sol. AG : sol. AZ :: ABCD} \times \text{AE : AMNO} \times \text{AX.}$$

Instead of the bases ABCD and AMNO, put $AB \times AD$ and $AO \times AM$; it will give

$$\text{sol. AG : sol. AZ :: } AB \times AD \times \text{AE : } AO \times AM \times \text{AX.}$$

Hence any two rectangular parallelopipedons are to each other, etc.

Scholium. We are consequently authorized to assume, as the measure of a rectangular parallelopipedon, the product of its base by its altitude; in other words, the product of its three dimensions.

In order to comprehend the nature of this measurement, it is necessary to reflect, that by the product of two or more lines is always meant the product of the numbers which represent them; those numbers themselves being determined by their linear unit, which may be assumed at pleasure. Upon this principle, the product of the three dimensions of a parallelopipedon is a

number, which signifies nothing of itself, and would be different if a different linear unit had been assumed; but if the three dimensions of another parallelopipedon are valued according to the same linear unit, and multiplied together in the same manner, the two products will be to each other as the solids, and will serve to express their relative magnitude.

The measure of a solid, or its magnitude, is called its volume. Hence we say the volume of a rectangular parallelopipedon is equal to the product of its base by its altitude, or to the product of its three dimensions.

As the cube has all its three dimensions equal, if the side is 1, the volume will be $1 \times 1 \times 1 = 1$; if the side is 2, the volume will be $2 \times 2 \times 2 = 8$; if the side is 3, the volume will be $3 \times 3 \times 3 = 27$; and so on. Hence, if the sides of a series of cubes are to each other as the numbers 1, 2, 3, etc., the cubes themselves (or their volumes) will be as the numbers 1, 8, 27, etc. Hence it is, that in arithmetic the *cube* of a number is the name given to the product which results from three factors, each equal to this number.

If it were proposed to find a cube double of a given cube, the side of the required cube would have to be to that of the given one, as the cube root of two is to unity. Now, by a geometrical construction, it is easy to find the square root of 2; but the cube root of it cannot be so found, at least not by the simple operations of elementary geometry, which consist in employing nothing but straight lines, two points of which are known, and circles whose centres and radii are determined.

Owing to this difficulty, the problem of *the duplication of the cube* became celebrated among the ancient geometers, as well as that of *the trisection of an angle*, which is nearly of the same species. The solutions of which such problems are susceptible, have, however, long since been discovered; and though less simple than the constructions of elementary geometry, they are not, on that account, less rigorous or less satisfactory.

THEOREM XII.

The volume of a parallelopipedon, and generally of any prism, is equal to the product of its base by its altitude.

For, in the first place, any parallelopipedon (T. VII.) is equivalent to a rectangular parallelopipedon having the same altitude and an equivalent base. Now the volume of the latter is equal to its base multiplied by its height; hence the volume of the former is, in like manner, equal to the product of its base by its altitude.

In the second place, and for a like reason, any triangular prism is half of the parallelopipedon so constructed as to have the same altitude and a double base; but the volume of the latter is equal to its base multiplied by its altitude; hence that of a triangular prism is also equal to the product of its base multiplied into its altitude.

In the third place, any prism may be divided into as many triangular prisms of the same altitude, as there are triangles capable of being formed in the polygon which constitutes its base; but the volume of each triangular prism is equal to its base multiplied by its altitude; and since the altitude is the same for all, it follows that the sum of all the partial prisms must be equal to the sum of all the partial triangles which constitute their bases, multiplied by the common altitude.

Hence the volume of any polygonal prism is equal to the product of its base by its altitude.

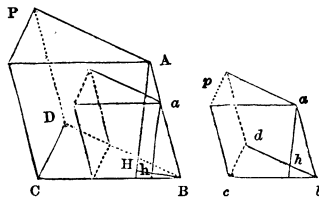
Cor. Comparing two prisms which have the same altitude, the products of their bases by their altitudes will be as the bases simply. Hence *two prisms of the same altitude are to each other as their bases.* For a like reason, *two prisms of the same base are to each other as their altitudes.*

THEOREM XIII.

Similar prisms are to one another as the cubes of their homologous sides.

Let P and p be two prisms, of which BC , bc are homologous sides: the prism P is to the prism p as the cube of BC to the cube of bc .

From A and a , homologous angles of the two prisms, draw



AH, ah perpendicular to the bases BCD, bcd : join BH; take $Ba = ba$, and in the plane BHA draw ah perpendicular to BH: then ah will be perpendicular to the plane CBD, and equal to ah the altitude of the other prism; for if the solid angles B and b were applied the one to the other, the planes which contain them, and consequently the perpendiculars ah , ah would coincide.

Now because of the similar triangles ABH, abh , and the similar figures P, p , we have

$$AH : ah :: AB : ab :: BC : bc;$$

and because of these similar bases, the

$$\text{base BCD} : \text{base } bcd :: BC^2 : bc^2. \quad (\text{B. III., T. XXVII.})$$

Taking the product of the corresponding terms of these proportions, we have

$$AH \times \text{base BCD} : ah \times \text{base } bcd :: BC^3 : bc^3.$$

But $AH \times \text{base BCD}$ expresses the volume of the prism P, and $ah \times \text{base } bcd$ expresses the volume of the other prism p ; therefore

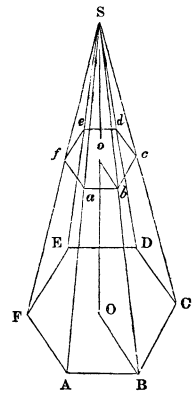
$$\text{prism P} : \text{prism } p :: BC^3 : bc^3.$$

THEOREM XIV.

If a pyramid be cut by a plane parallel to its base, the section thus formed will be a polygon similar to the base, and the lateral edges and the altitude will be cut into proportional parts.

First, since the planes ABCDEF, $abcdef$ are parallel, it follows that AB and ab , BC and bc , CD and cd , etc., are parallel (B. V., T. X.); consequently the angles ABC and abc , BCD and bcd are respectively equal (B. V., T. XIII.). Moreover the couples of similar triangles SAB and Sab , SBC and Sbc , etc., give the following equal ratios:

$$\begin{array}{lll} SA : Sa :: AB : ab :: SB : Sb, \\ SB : Sb :: BC : bc :: SC : Sc, \\ SC : Sc :: CD : cd :: SD : Sd, \\ \&c. \quad \&c. \quad \&c. \end{array}$$



Hence, by equality of ratios, we have

$$AB : ab :: BC : bc :: CD : cd :: \text{etc.}$$

Consequently the two polygons $ABCDEF$, $abcdef$ are similar (B. III., T. X.).

Now, considering only the ratios between the lateral edges, we have

$$SA : Sa :: SB : Sb :: SC : Sc :: \text{etc.},$$

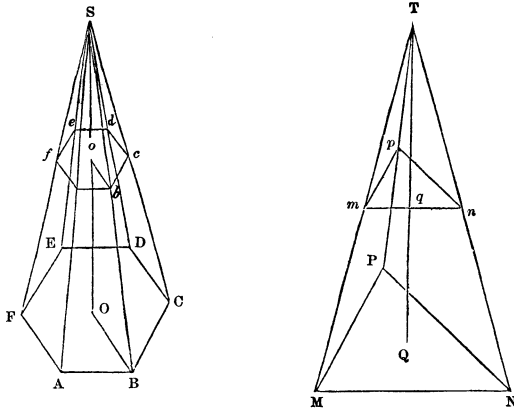
from which we see that the edges are cut into proportional parts.

Finally, if we pass a plane through the edge SB and the altitude SO , its intersection bo with the plane $abcde$ will be parallel to BO (B. V., T. X.), and the two similar triangles SBO , Sbo will give

$$SO : So :: SB : Sb :: SA : Sa :: \text{etc.},$$

which establishes the theorem.

Cor. When two pyramids $S - ABCDE$, $T - MNP$ have equivalent bases situated in the same plane, and equal altitudes, the sections made by a plane parallel to their bases, are also equivalent.



For, since the polygons $ABCDE$, $abcde$ are similar, we have the proportion

$$ABCDE : abcde :: AB^2 : ab^2;$$

we also have, by reason of the foregoing relations,

$$ABCDE : abcde :: SO^2 : So^2.$$

In the same manner the two polygons MNP , mnp give

$$MNP : mnp :: TQ^2 : Tq^2,$$

we have, moreover, by supposition, $SO = TQ$, $So = Tq$; hence,

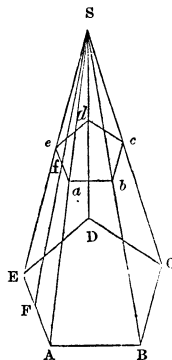
$$ABCDE : abcde :: MNP : mnp.$$

Now, by hypothesis, $ABCDE = MNP$; consequently, $abcde = mnp$.

THEOREM XV.

The lateral surface of a right pyramid is equal to the perimeter of its base multiplied by half the slant height.

Since the lateral surface is composed of equal isosceles triangles SAB , SBC , SCD , etc., each one of which is measured by its base into one half its altitude, which altitude is the same as SF the slant height of the pyramid, it follows that the lateral surface is equal to the sum of the bases of all these triangles, or the perimeter of the pyramid's base, into one half the slant height.



Cor. I. If two right pyramids have the same slant heights, their lateral surfaces will be to each other as the perimeters of their bases.

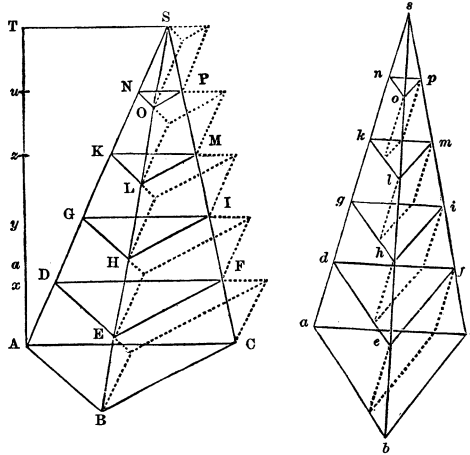
Cor. II. The lateral surface of a frustum of a regular pyramid is measured by half the sum of the perimeters of its two bases multiplied by its slant height.

For, since the frustum is formed from a regular pyramid, the two bases are similar and regular polygons (T. XIV.). Consequently the lateral surface of a frustum of a right pyramid is composed of equal trapezoids, the altitude of each being the same as the slant height of the frustum.

The area of each trapezoid is measured by half the sum of its parallel bases multiplied into its altitude (B. III., T. XXIII., C. II.). Hence the sum of all these trapezoids, or the lateral surface of the frustum, is measured by half the sum of the perimeters of its bases into its slant height.

THEOREM XVI.

Two triangular pyramids having equivalent bases and equal altitudes, are equivalent, or equal in volume.



Let $S-ABC$, $S-abc$ be those two pyramids; let their equivalent bases ABC , abc be situated in the same plane, and let AT be their common altitude. If they are not equivalent, let $S-abc$ be the smaller; and suppose Aa to be the altitude of a prism, which, having ABC for its base, is equal to their difference.

Divide the altitude AT into equal parts Ax , xy , yz , etc., each less than Aa , and let k be one of those parts; through the points of division, pass planes parallel to the plane of the bases: the corresponding sections formed by these planes in the two pyramids will be respectively equivalent (T. XIV., C.), namely, DEF to def , GHI to ghi , etc.

This being granted, upon the triangles ABC , DEF , GHI , etc., taken as bases, construct exterior prisms, having for edges the parts AD , DG , GK , etc., of the edge SA . In like manner, on the bases def , ghi , klm , etc., in the second pyramid, construct interior prisms, having for edges the corresponding parts of sa . It is plain that the sum of all the exterior prisms of the pyramid $S-ABC$ will be greater than this pyramid; and, also, that the sum of all the interior prisms of the pyramid $s-abc$ will be

less than this. Hence the difference between the sum of all the exterior prisms and the sum of all the interior ones, must be greater than the difference between the two pyramids themselves.

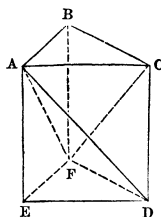
Now, beginning with the bases ABC , abc , the second exterior prism $DEFG$ is equivalent to the first interior prism $defa$, because they have the same altitude k , and their bases DEF , def are equivalent: for like reasons, the third exterior prism $GHIK$, and the second interior prism $ghid$, are equivalent; the fourth exterior, and the third interior; and so on, to the last in each series. Hence all the exterior prisms of the pyramid $S - ABC$, excepting the first prism $DABC$, have equivalent corresponding ones in the interior prisms of the pyramid $s - abc$: hence the prism $DABC$ is the difference between the sum of all the exterior prisms of the pyramid $S - ABC$, and the sum of all the interior prisms of the pyramid $s - abc$. But the difference between these two sets of prisms has already been proved to be greater than that of the two pyramids, which latter difference we supposed to be equal to the prism $aABC$: hence the prism $DABC$ must be greater than the prism $aABC$; but in reality it is less, for they have the same base ABC , and the altitude Ax of the first is less than Aa the altitude of the second. Hence the supposed inequality between the two pyramids cannot exist; hence the two pyramids $S - ABC$, $s - abc$, having equal altitudes and equivalent bases, are themselves equivalent.

THEOREM XVII.

Every triangular pyramid is the third of the triangular prism having the same base and altitude.

Let $F - ABC$ be a triangular pyramid, $ABCDEF$ a triangular prism of the same base and altitude; the pyramid will be equal to one third of the prism.

Conceive the pyramid $F - ABC$ to be cut off from the prism by a section made along the plane FAC , and there will remain the solid $FACDE$, which may be considered as a quadrangular pyramid whose vertex is F , and base the parallelogram $ACDE$. Draw



the diagonal AD , and extend the plane FAD , which will cut the quadrangular pyramid into two triangular ones $F-ACD$, $F-ADE$. These two triangular pyramids have for their common altitude the perpendicular drawn from F to the plane $ACDE$; they have equal bases, the triangles ACD , ADE being halves of the same parallelogram; hence the two pyramids $F-ACD$, $F-ADE$ are equivalent. But the pyramid $F-ADE$ and the pyramid $F-ABC$ have equal bases, ABC , DEF ; they have also the same altitude, namely, the distance of the parallel planes ABC , DEF : hence the two pyramids are equivalent. Now the pyramid $F-ADE$ has already been proved equivalent to $F-ACD$; hence the three pyramids $F-ABC$, $F-ADE$, $F-ACD$, which compose the prism $ABCD$, are all equivalent. Hence the pyramid $F-ABC$ is the third part of the prism $ABCD$, which has the same base and the same altitude.

Cor. The volume of a triangular pyramid is equal to a third part of the product of its base by its altitude.

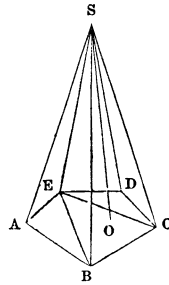
THEOREM XVIII.

The volume of any pyramid has for its measure the area of its base multiplied into one third of its altitude.

Let $S-ABCDE$ be a pyramid, having the altitude SO ; then will it be measured by the base $ABCDE$ into one third of the altitude SO .

For, extending the planes SEB , SEC through the diagonals EB , EC , the polygonal pyramid $S-ABCDE$ will be divided into several triangular pyramids, all having the same altitude SO . But (T. XVII., C.) each of these pyramids is measured by multiplying its base ABE , BCE , or CDE by the third part of its altitude SO ; hence the sum of these triangular pyramids, or the polygonal pyramid $S-ABCDE$ will be measured by the sum of the triangles ABE , BCE , CDE , or the polygon $ABCDE$, multiplied by $\frac{1}{3}SO$. Hence every pyramid is measured by a third part of the product of its base by its altitude.

Cor. I. Every pyramid is the third part of the prism which has the same base and the same altitude.



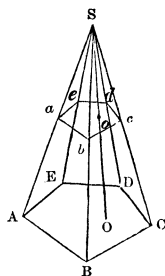
Cor. II. Two pyramids having the same altitude, are to each other as their bases.

Scholium. The volume of any polyedral body may be computed, by dividing the body into pyramids; and this division may be accomplished in various ways. One of the simplest is to make all the planes of division pass through the vertex of one solid angle; in that case, there will be formed as many partial pyramids as the polyedron has faces, *minus* those faces which form the polyedral angle whence the planes of division proceed.

THEOREM XIX.

The volume of a frustum of a pyramid is equivalent to three pyramids having the common altitude of the frustum, and for bases, the lower base of the frustum, the upper base, and a mean proportional between them.

We will represent the lower base ABCDE of the frustum by A , and the upper base $abcde$ by a ; also, we will denote the altitude oO of the frustum by h . If we denote the altitude SO of the pyramid, whose base is ABCDE by x , we shall have $x - h$ for the altitude So of the pyramid whose base is $abcde$.



Since the two bases of the frustum are similar polygons (T. XIV.), their areas are to each other as the squares of their homologous sides AB and ab (B. III., T. XXVIII.), which sides are to each other as SO to So ; consequently we have

$$A : a :: x^2 : (x - h)^2.$$

Extracting the square root of each term, we have

$$A^{\frac{1}{2}} : a^{\frac{1}{2}} :: x : x - h.$$

This proportion immediately gives $x = \frac{A^{\frac{1}{2}}}{A^{\frac{1}{2}} - a^{\frac{1}{2}}} \times h$, and con-

sequently, $x - h = \frac{a^{\frac{1}{2}}}{A^{\frac{1}{2}} - a^{\frac{1}{2}}} \times h$.

Hence the volume of the larger pyramid is $A \times \frac{1}{3}x = \frac{A^{\frac{3}{2}}}{A^{\frac{1}{2}} - a^{\frac{1}{2}}} \times \frac{1}{3}h$. The volume of the smaller pyramid is

$$a \times \frac{1}{3}(x - h) = \frac{a^{\frac{3}{2}}}{A^{\frac{1}{2}} - a^{\frac{1}{2}}} \times \frac{1}{3}h.$$

If we denote the volume of the frustum, which is the difference of these pyramids by V , we shall have $V = \frac{A^{\frac{3}{2}} - a^{\frac{3}{2}}}{A^{\frac{1}{2}} - a^{\frac{1}{2}}} \times \frac{1}{3}h$, which becomes $V = (A + a + A^{\frac{1}{2}}a^{\frac{1}{2}}) \times \frac{1}{3}h = A \times \frac{1}{3}h + a \times \frac{1}{3}h + \sqrt{A \times a} \times \frac{1}{3}h$, which establishes the Theorem.

THEOREM XX.

Two similar pyramids are to each other as the cubes of their homologous sides.

For, two pyramids being similar, the smaller may be placed within the greater, so that the angle S shall be common to both. In that position the bases $ABCDE$, $abcde$ will be parallel; because, since the homologous faces are similar, the angle Sab is equal to SAB , and Sbc to SBC ; hence the plane ABC is parallel to the plane abc . This granted, let SO be the perpendicular drawn from the vertex S to the plane ABC , and o the point where this perpendicular meets the plane abc : from what has already been shown (T. XIV.), we shall have

$$SO : So :: SA : Sa :: AB : ab;$$

and, consequently, $\frac{1}{3}SO : \frac{1}{3}So :: AB : ab$.

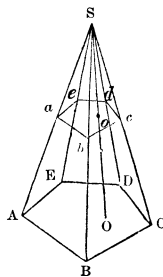
But the bases $ABCDE$, $abcde$ being similar figures, we have

$$ABCDE : abcde :: AB^2 : ab^2.$$

Multiply the corresponding terms of these two proportions; there results the proportion,

$$ABCDE \times \frac{1}{3}SO : abcde \times \frac{1}{3}So :: AB^3 : ab^3.$$

Now $ABCDE \times \frac{1}{3}SO$ is the volume of the pyramid $S-ABCDE$, and $abcde \times \frac{1}{3}So$ is that of the pyramid $S-abcde$ (T. XVII., C.); hence two similar pyramids are to each other as the cubes of their homologous sides.



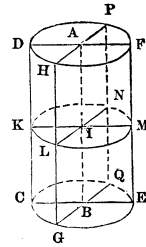
SEVENTH BOOK.

THE THREE ROUND BODIES.

DEFINITIONS.

I. A *cylinder* is a solid, which may be produced or generated by the revolution of a rect-angle $ABCD$, conceived to revolve about the side AB .

In this rotation, the sides AD , BC , continuing always perpendicular to AB , describe equal circular planes DHP , CGQ , which are called *the bases of the cylinder*; the side CD at the same time describing *the convex surface*.



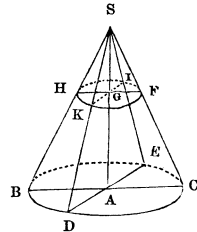
The immovable line AB is called *the axis of the cylinder*.

Every section KLM made in the cylinder, at right angles to the axis, is a circle equal to either of the bases; for, while the rectangle $ABCD$ revolves about AB , the line KI , perpendicular to AB , describes a circular plane, equal to the base, which is a section made perpendicular to the axis at the point I .

Every section $PQGH$ passing through the axis is a rectangle, and is double of the generating rectangle $ABCD$.

II. A *cone* is a solid, which may be produced or generated by the revolution of a right-angled triangle SAB , conceived to revolve about the side SA .

In this rotation, the side AB describes a circular plane $BDCE$, named *the base of the cone*; and the hypotenuse SB describes its *convex surface*.



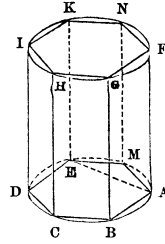
The point S is named *the vertex of the cone*; SA its *axis*, or *altitude*.

Every section HKFI formed at right angles to the axis, is a circle. Every section SDE passing through the axis, is an isosceles triangle double of the generating triangle SAB.

III. If, from the cone SCDB, the cone SFKH be cut off by a section parallel to the base, the remaining solid CBHF is called *a truncated cone*, or *the frustum of a cone*. We may conceive it to be described by the revolution of a trapezium ABHG, whose angles A and C are right, about the side AG. The immovable line AG is called *the axis* or *altitude of the frustum*; the circles BDC, HFK are its *bases*, and BH is its *slant height*.

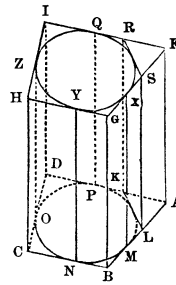
IV. Two cylinders or two cones, are *similar*, when their axes are to each other as the diameters of their bases.

V. If, in the circle ACD which forms the base of a cylinder, a polygon ABCDEM be inscribed, a right prism, constructed on this base ABCDEM, and equal in altitude to the cylinder, is said to be *inscribed in the cylinder*, or the cylinder to be *circumscribed about the prism*.



The edges AF, BG, CH, etc., of the prism, being perpendicular to the plane of the base, are evidently included in the convex surface of the cylinder. Hence the prism and the cylinder touch one another along these edges.

VI. In like manner, if ABCD is a polygon circumscribed about the base of a cylinder, a right prism, constructed on this base ABCD, and equal in altitude to the cylinder, is said to be *circumscribed about the cylinder*, or the cylinder to be *inscribed in the prism*.



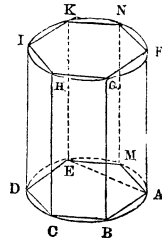
Let M, N, etc., be the points of contact in the sides AB, BC, etc.; and through the points M, N, etc., let MX, NY, etc., be drawn perpendicular to the plane of the base: those perpendiculars will evidently lie both in the surface of the cylinder, and in that of the circumscribed prism; hence they will be their lines of contact.

VII. A *sphere* is a solid terminated by a curved surface, all the points of which are equally distant from a point within, called the *centre*.

THEOREM I.

The lateral or convex surface of a cylinder has for its measure the product of its circumference into its altitude.

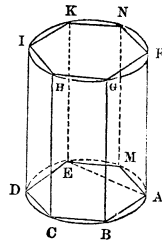
In the cylinder, suppose a right prism to be inscribed, having a regular polygon for its base. The lateral surface of this prism has for its measure the perimeter of its base multiplied by its altitude (B. VI., T. I.). When the number of sides of the polygon forming the base of the inscribed prism is indefinitely increased, its limit will become the circle constituting the base of the cylinder (B. IV., T. VIII., S.). We may, therefore, regard a cylinder as a right prism having a regular polygon of an infinite number of infinitely small sides for its base; and since the lateral surface of a right prism will always have for its measure the perimeter of its base into its altitude, it follows that the lateral surface of a cylinder has for its measure the circumference of its base into its altitude.



THEOREM II.

The volume of a cylinder has for its measure the product of its base into its altitude.

As in the last Theorem, if we regard a cylinder as a right prism, having a regular polygon of an infinite number of sides for its base, and recall to mind that the volume of a right prism is the product of its base into its altitude (B. VI., T. XII.), we shall at once see that the volume of a cylinder is equal to the product of its base into its altitude.



Cor. I. Cylinders of the same altitude are to each other as their bases; and cylinders of the same base are to each other as their altitudes.

Cor. II. Similar cylinders are to each other as the cubes of their altitudes, or as the cubes of the diameters of their bases. For the bases are as the squares of their diameters; and the cylinders being similar, the diameters of their bases (D. IV.) are to

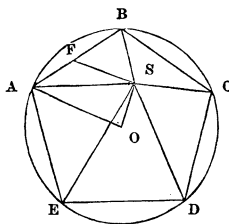
each other as the altitudes: hence the bases are as the squares of the altitudes; consequently, the bases multiplied by the altitudes, or the cylinders themselves, are as the cubes of the altitudes.

Scholium. Let R be the radius of a cylinder's base, and H the altitude. The area of the base (B. IV., T. XIV., S.) will be πR^2 ; and the volume of the cylinder will be $\pi R^2 \times H$, or $\pi R^2 H$, where $\pi = 3.141592$, etc.

THEOREM III.

The convex, or lateral surface of a cone, is equal to the product of the circumference of its base into half its slant height.

Let the circle whose radius is OA be the base of the cone, S its vertex, and SA its slant height. Then will the convex surface of the cone have for its measure *circ.* $OA \times \frac{1}{2}SA$.



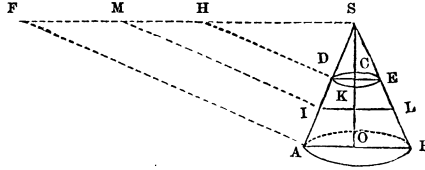
For, conceive a regular polygon inscribed in the circle OA ; and on this polygon, as a base, a pyramid having S for its vertex, to be constructed. The lateral surface of this pyramid will have for its measure the perimeter of the polygon, constituting its base, into one half of SF its slant height (B. VI., T. XV.). When the number of sides of the inscribed polygon is indefinitely increased, its perimeter will be *limited* by the circumference of the circle, its slant height will be *limited* by the slant height of the cone, and the *limit* of the lateral surface will be the convex surface of the cone. A cone may thus be regarded as a right pyramid, having a regular polygon of an infinite number of infinitely short sides for its base. And since the lateral surface of a pyramid will always have for its measure the perimeter of its base into half its slant height, however great may be the number of sides in the polygon forming its base, it follows, that the convex surface of a cone has for its measure the circumference of its base into half its slant height.

Scholium. Let L be the slant height of a cone, R the radius of its base. The circumference of this base will be $2\pi R$; and the convex surface of the cone will be $2\pi R \times \frac{1}{2}L$, or πRL .

THEOREM IV.

The convex surface of a truncated cone is equal to its side multiplied by half the sum of the circumferences of its two bases.

In the plane SAB which passes through the axis SO, draw the line AF perpendicular to SA, and equal to the circumference having AO for its radius; join SF, and draw DH parallel to AF.



From the similar triangles SAO, SDC, we have

$$AO : DC :: SA : SD;$$

and by the similar triangles SAF, SDH,

$$AF : DH :: SA : SD;$$

hence,

$$AF : DH :: AO : DC;$$

or (B. IV., T. XIV.), as circ. AO is to circ. DC. But, by construction, $AF = \text{circ. AO}$; hence $DH = \text{circ. DC}$. Hence the triangle SAF, measured by $AF \times \frac{1}{2}SA$, is equal to the surface of the cone SAB, which is measured by $\text{circ. AO} \times \frac{1}{2}SA$. For a like reason, the triangle SDH is equal to the surface of the cone SDE. Therefore the surface of the truncated cone ADEB is equal to that of the trapezium ADHF; but the latter is measured by $AD \times \left(\frac{AF + DH}{2} \right)$ (B. III., T. XXIII., C. II.).

Hence the surface of the truncated cone ADEB is equal to its side AD multiplied by half the sum of the circumferences of its two bases.

Scholium. If a line AD, lying wholly on one side of the line OC, and in the same plane, make a revolution around OC, the surface described by AD will have for its measure

$$AD \times \left(\frac{\text{circ. AO} + \text{circ. DC}}{2} \right),$$

the lines AO, DC being perpendiculars, drawn from the extremities of the axis OC.

For, if AD and OC are produced till they meet in S, the surface described by AD is evidently that of a truncated cone having AO and DC for the radii of its bases, the vertex of the whole cone being S. Hence this surface will be measured as above.

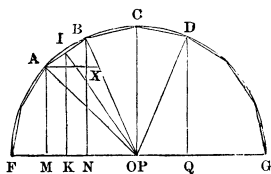
This measure will always hold good, even when the point D falls on S, and thus forms a whole cone; and also when the line AD is parallel to the axis, and thus forms a cylinder. In the first case, DC would be nothing; in the second, DC would be equal to AO and to IK.

Cor. I. Through I, the middle point of AD, draw IKL parallel to AB, and IM parallel to AF: it may be shown, as above, that $IM = \text{circ. IK}$; but the

$$\text{trapezium ADHF} = AD \times IM = AD \times \text{circ. IK}.$$

Hence it may also be asserted, that *the surface of a truncated cone is equal to its side multiplied by the circumference of a section at equal distances from the two bases.*

Cor. II. The point I being the middle of AB, and IK a perpendicular drawn from the point I to the axis, the surface described by AB, by the last Corollary, will have for its measure $AB \times \text{circ. IK}$. Draw AX parallel to the axis; the triangles ABX, OIK will have their sides perpendicular each to each, namely, OI to AB, IK to AX, and OK to BX; hence these triangles are similar, and give the proportion



$$AB : AX \text{ or } MN :: OI : IK,$$

or as circ. OI to circ. IK; hence $AB \times \text{circ. IK} = MN \times \text{circ. OI}$. Whence it is plain that the surface described by the partial polygon ABCD is measured by $(MN + NP + PQ) \times \text{circ. OI}$, or by $MQ \times \text{circ. OI}$; hence it is equal to the altitude multiplied by the circumference of the inscribed circle.

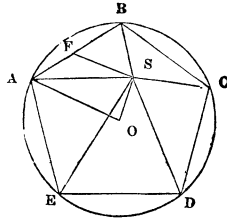
Cor. III. If the whole polygon has an even number of sides, and if the axis FG passes through two opposite vertices F and G, the whole surface described by the revolution of the half polygon FACG will be equal to its axis FG multiplied by the circumference of the inscribed circle. This axis FG will, at the same time, be the diameter of the circumscribed circle.

THEOREM V.

The volume of a cone has for its measure the product of its base into one third of its altitude.

Let the cone have OA for the radius of its base, and SO for its altitude; then will its volume be represented by *area* $OA \times \frac{1}{3}SO$.

As in Theorem III., conceive a regular pyramid to be inscribed in the cone. The volume of this pyramid will have for its measure the area of the polygon constituting its base multiplied by one third of its altitude (B. VI., T. XVIII.). When the number of sides of the polygon forming the base of the inscribed pyramid is indefinitely increased, the area of the polygon will have for its *limit* the area of the circle forming the base of the cone, and the *limit* of the inscribed pyramid will be the cone. Hence, the volume of a cone has for its measure the product of its base into one third of its altitude.



Cor. I. A cone is the third of a cylinder having the same base and the same altitude. Whence it follows:

1. That cones of equal altitudes are to each other as their bases;
2. That cones of equal bases are to each other as their altitudes;
3. That similar cones are as the cubes of the diameters of their bases, or as the cubes of their altitudes.

Scholium. If R be the radius of a cone's base, and H its altitude, the volume of the cone will be $\pi R^2 \times \frac{1}{3}H$, or $\frac{1}{3}\pi R^2 H$.

Cor. II. If a plane be drawn parallel to the base of a cone, cutting it so as to form a frustum of a cone, it will at the same time cut the inscribed pyramid, forming also a frustum of a regular pyramid inscribed in the frustum of the cone. The *limit* of the inscribed frustum will be the frustum of the cone. And since the volume of the frustum of a pyramid is equivalent to the volumes of three pyramids having the common altitude of the frustum, and for bases the lower base of the frustum, the upper base of the frustum, and a mean proportional between them (B. VI., T. XIX.), it follows, that *the volume of the frustum*

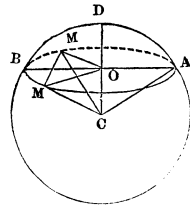
of a cone is equivalent to three cones having the common altitude of the frustum, and for bases the lower base of the frustum, the upper base, and a mean proportional between them.

∧ If the radii of the two bases of a frustum of a cone be represented by R and R' , its altitude by h , and its volume by V , we shall have $V = \frac{1}{3}\pi h(R^2 + R'^2 + RR')$.

THEOREM VI.

Every section of a sphere, made by a plane, is a circle.

Let AMB be the section, made by a plane in the sphere whose centre is C . From the point C draw CO perpendicular to the plane AMB ; and draw lines CM , CM to different points of the curve AMB , which terminates the section.



The oblique lines CM , CM , CA being equal, being radii of the sphere, they are equally distant from the perpendicular CO (B. V., T. V.); hence all the lines OM , OM , OB are equal; hence the section AMB is a circle, whose centre is O .

Cor. I. If the section passes through the centre of the sphere, its radius will be the radius of the sphere; hence all great circles are equal.

Cor. II. Two great circles always bisect each other; for their common intersection, passing through the centre, is a diameter.

Cor. III. Every great circle divides the sphere and its surface into two equal parts; for, if the two hemispheres were separated, and afterwards placed on the common base, with their convexities turned the same way, the two surfaces would exactly coincide, no point of the one being nearer the centre than any point of the other.

Cor. IV. The centre of a small circle, and that of the sphere, are in the same straight line perpendicular to the plane of the little circle.

Cor. V. Small circles are the less the further they lie from the centre of the sphere; for, the greater CO is, the less is the chord AB , the diameter of the small circle AMB .

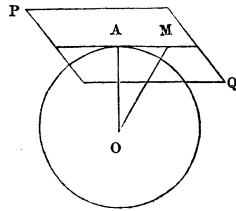
Cor. VI. An arc of a great circle may always be made to pass

through any two given points in the surface of the sphere; for the two given points and the centre of the sphere make three points, which determine the position of a plane. But if the two given points were at the extremities of a diameter, these two points and the centre would then lie in one straight line, and an infinite number of great circles might be made to pass through the two given points.

THEOREM VII.

A plane perpendicular to the radius at its extremity, is tangent to the sphere.

Let PQ be a plane perpendicular to the radius OA; at its extremity, A, it will be tangent to the sphere.

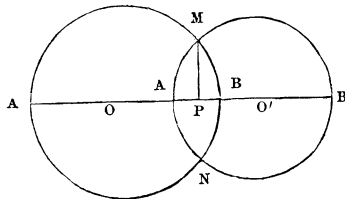


For, taking any other point, as M, in this plane, if we draw OM, it will be longer than the perpendicular OA (B. V., T. V.); consequently the point M is situated without the surface of the sphere. Hence the plane PQ can have only the point A common with this surface; it is therefore tangent to the sphere (D. IX.).

THEOREM VIII.

When two spherical surfaces cut each other, the line of intersection is a circumference of a circle, of which the plane is perpendicular to the line of their centres, and whose centre is situated on this line.

Let O, O' be the centres of two spheres, M a point common to their surfaces, and MP a perpendicular drawn to the line of their centres. If we draw a plane through the three points O, O', M, it will intersect the



surface of the two spheres in arcs of great circles. Now, suppose the two semicircles AMB, A'MB' to make a complete revolution about the common line AB' as an axis, it is evident that the per-

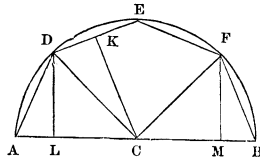
pendicular MP will describe a circle common to the two spheres thus engendered. From which we see that the surface of the spheres cut each other in the circumference of a circle, whose radius is the perpendicular drawn from the point M to this axis.

Scholium. Whatever may be the relative positions of two spheres, since all planes passing through the line of their centres give two circumferences whose centres and whose radii are those of the spheres themselves, it follows that the condition of *contact* and of *intersection* of their surfaces are, in all respects, identical with those in reference to the two circumferences. So that to enumerate and demonstrate these different conditions, it is sufficient to refer to Book II., Theorems XVIII., XIX., XX., and XXI. We will confine ourselves in this place to noticing the condition relative to *contact*: *When two spheres touch each other, the distance between their centres is equal to the sum or the difference of their radii. They have at their point of contact a common tangent plane.*

THEOREM IX.

The surface of a sphere is equal to its diameter multiplied by the circumference of a great circle.

Let the surface of the sphere be engendered by the revolution of the semi-circumference ADEFB about its diameter AB; it will have for its measure its diameter multiplied by its entire circumference.



Suppose a regular semi-polygon to be inscribed in the semicircle, and to revolve simultaneously with the semicircle; then will the surface engendered by the semi-polygon have for its measure the diameter AB multiplied by the circ. of its apothem CK (T. IV., C. III.). Now, when the number of sides of this inscribed semi-polygon is indefinitely increased, the *limit* of its perimeter will become the semi-circumference, its apothem will become the radius of the semicircle, and the surface engendered by the revolution of this semi-polygon will become the surface of the sphere. Hence, the surface of a sphere has for its measure the product of its diameter into its circumference.

Cor. I. The surface of the great circle is measured by multiplying its circumference by half the radius, or by a fourth of the diameter; hence, *the surface of a sphere is four times that of its great circle.*

Cor. II. *The surface of a zone has for its measure its altitude multiplied by the circumference of a great circle.*

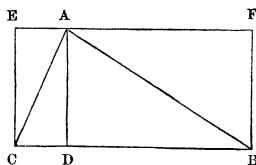
For, the surface described by any portion of the inscribed polygon, as $DE + EF$, has for its measure LM into the circ. of its apothem CK (T. IV., C. II.); which, at the *limit*, gives the surface of the zone equal to its altitude LM multiplied into its circumference CA .

Cor. III. Two zones, taken in the same sphere, or in equal spheres, are to each other as their altitudes; and any zone is to the surface of the sphere, as the altitude of that zone is to the diameter of the sphere.

THEOREM X.

If a triangle and a rectangle, having the same base and the same altitude, turn simultaneously about the common base, the solid described by the revolution of the triangle will be a third of the cylinder described by the revolution of the rectangle.

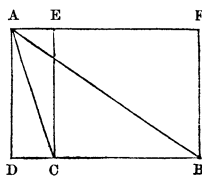
Let ABC be the triangle, and EB the rectangle.



To the axis draw the perpendicular AD ; the cone described by the triangle ABD is the third part of the cylinder described by the rectangle $AFBD$ (T. V., C. I.); also the cone described by the triangle ADC is the

third part of the cylinder described by the rectangle $ADCE$: hence the sum of the two cones, or the solid described by ABC , is the third part of the two cylinders taken together, or of the cylinder described by the rectangle $BCEF$.

If the perpendicular AD falls without the triangle, the solid described by ABC will be the difference of the two cones described by ABD and ACD ; but, at the same time, the cylinder described by $BCEF$ will be the difference of the two cylinders described by



AFBD and AECD. Hence the solid described by the revolution of the triangle will still be a third part of the cylinder described by the revolution of the rectangle having the same base and altitude.

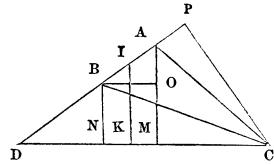
Scholium. The circle of which AD is radius has for its measure $\pi \times AD^2$; hence $\pi \times AD^2 \times BC$ measures the cylinder described by BCEF, and $\frac{1}{3}\pi \times AD^2 \times BC$ measures the solid described by the triangle ABC.

THEOREM XI.

If a triangle be revolved about a line drawn at pleasure through its vertex, the solid described by the triangle will have for its measure the area of the triangle multiplied by two thirds of the circumference traced by the middle point of the base.

Let CAB be the triangle, and CD the line about which it revolves.

Produce the side AB till it meets the axis CD in D; from the points A and B, draw AM, BN perpendicular to the axis.



The solid described by the triangle CAD is measured (T. X., S.) by $\frac{1}{3}\pi \times AM^2 \times CD$; the solid described by the triangle CBD is measured by $\frac{1}{3}\pi \times BN^2 \times CD$: hence the difference of those solids, or the solid described by ABC, will have for its measure $\frac{1}{3}\pi (AM^2 - BN^2) \times CD$.

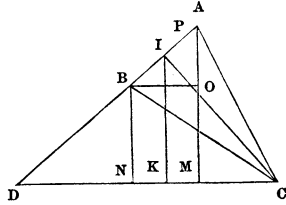
To this expression another form may be given. From I, the middle point of AB, draw IK perpendicular to CD; and through B draw BO parallel to CD: we shall have $AM + BN = 2IK$, and $AM - BN = AO$; hence $(AM + BN) \times (AM - BN)$, or $AM^2 - BN^2 = 2IK \times AO$. Hence the measure of the solid in question is expressed by $\frac{2}{3}\pi \times IK \times AO \times CD$. But if CP is drawn perpendicular to AB, the triangles ABO, DCP will be similar, and give the proportion,

$$AO : CP :: AB : CD;$$

hence $AO \times CD = CP \times AB$, but $CP \times AB$ is double the area of the triangle ABC; hence we have $AO \times CD = 2ABC$; hence the solid described by the triangle ABC is also measured by

$\frac{4}{3}\pi \times ABC \times IK$, or, which is the same thing, by $ABC \times \frac{2}{3} \text{ circ. IK}$, circ. IK being equal to $2\pi \times IK$. Hence, *the solid described by the revolution of the triangle ABC has for its measure the area of this triangle multiplied by two thirds of the circumference traced by I, the middle point of the base.*

Cor. I. If the side $AC = CB$, the line CI will be perpendicular to AB , the area ABC will be equal to $AB \times \frac{1}{2} CI$, and the volume $\frac{4}{3}\pi \times ABC \times IK$ will become $\frac{2}{3}\pi \times AB \times IK \times CI$. But the triangles ABO , CIK are similar, and give the proportion,

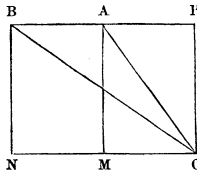


$$AB : BO \text{ or } MN :: CI : IK;$$

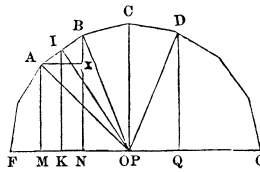
hence $AB \times IK = MN \times CI$: hence the solid described by the isosceles triangle ABC will have for its measure $\frac{2}{3}\pi \times MN \times CI^2$.

Cor. II. The general solution appears to include the supposition that AB produced will meet the axis; but the results would be equally true though AB were parallel to the axis.

Thus the cylinder described by $AMNB$ is equal to $\pi \cdot AM^2 \cdot MN$; the cone described by ACM is equal to $\frac{1}{3}\pi AM^2 \cdot CM$, and the cone described by BCN to $\frac{1}{3}\pi \cdot AM^2 \cdot CN$. Add the first two solids, and take away the third: we shall have the measure of the solid described by ABC equal to $\pi \cdot AM^2 \cdot (MN + \frac{1}{3}CM - \frac{1}{3}CN)$; and since $CN - CM = MN$, this expression is reducible to $\pi \cdot AM^2 \cdot \frac{2}{3}MN$, or $\frac{2}{3}\pi CP^2 \cdot MN$, which agrees with the conclusion above drawn.



Cor. III. Let AB , BC , CD be the several successive sides of a regular polygon, O its centre, and OI the radius of the inscribed circle; if the polygonal sector AOD lying all on one side of the diameter FG be supposed to perform a revolution about this diameter, the solid so described will have for its measure $\frac{2}{3}\pi \cdot OI^2 \cdot MQ$, MQ being that portion of the axis which is included by the extreme perpendiculars AM , DQ .



For, since the polygon is regular, all the triangles, AOB , BOC ,

etc., are equal and isosceles. Now, by Corollary I., the solid produced by the isosceles triangle AOB has for its measure $\frac{2}{3}\pi \cdot OI^2 \cdot MN$; the solid described by the triangle BOC has for its measure $\frac{2}{3}\pi \cdot OI^2 \cdot NP$, and the solid described by the triangle COD has for its measure $\frac{2}{3}\pi \cdot OI^2 \cdot PQ$: hence the sum of those solids, or the whole solid described by the polygonal sector AOD will have for its measure $\frac{2}{3}\pi \cdot OI^2 \cdot (MN + NP + IQ)$, or $\frac{2}{3}\pi \cdot OI^2 \cdot MQ$.

Cor. IV. The solid generated by the revolution of the entire polygon will have for its measure $\frac{2}{3}\pi \times OI^2 \times FG$. But the surface generated by the entire polygon has for its measure $2\pi \times OI \times FG$ (T. IV., C. III.), since $2\pi \times OI$ is the circumference of the inscribed circle. Hence the solid is measured by $2\pi \times OI \times FG$ into $\frac{1}{3}OI$; that is, the solid generated by the revolution of a semi-polygon inscribed in a semicircle, has for its measure its surface multiplied by one third the apothem of the generating polygon.

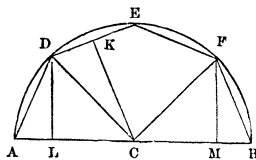
THEOREM XII.

The volume of a sphere is measured by its surface multiplied by one third its radius.

As in Theorem IX., let the sphere be generated by the revolution of the semicircle whose radius is CA, about its diameter AB. The regular inscribed polygon will generate a solid having for its measure the surface of the solid multiplied by one third the apothem of the polygon (T. XI., C. IV.). When we pass to the *limit*, the surface of the solid generated by the polygon will be the surface of the sphere generated by the semicircle (T. IX.), and the apothem of the polygon will become the radius. Consequently, the volume of a sphere is measured by its surface multiplied into one third of its radius.

Cor. The volume of a spherical sector has for its measure the zone which forms its base multiplied into one third of its radius.

For, the solid generated by any sectorial portion of the inscribed polygon, as CDEF, has for its measure $\frac{2}{3}\pi \times CK^2 \times LM$



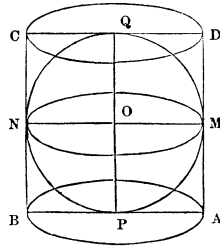
(T. XI., C. III.). Hence at the limit we have, for the measure of the spherical sector generated by the plane sector CDF, $\frac{2}{3}\pi \times CA^2 \times LM' = 2\pi \times CA \times LM \times \frac{1}{3}CA$. But $2\pi \times CA \times LM$ is the measure of the surface of the zone forming the base of the spherical sector; hence the volume of a spherical sector is measured by its surface into one third of its radius.

Scholium. Let R be the radius of a sphere: its surface will be $4\pi R^2$; its solidity, $4\pi R^2 \times \frac{1}{3}R$, or $\frac{4}{3}\pi \cdot R^3$. If the diameter is named D , we shall have $R = \frac{1}{2}D$, and $R^3 = \frac{1}{8}D^3$: hence the solidity may likewise be expressed by $\frac{4}{3}\pi \cdot \frac{1}{8}D^3$, or $\frac{1}{6}\pi D^3$.

THEOREM XIII.

The surface of a sphere is to the whole surface of the circumscribed cylinder (including its bases), as 2 is to 3; and the volumes of these two bodies are to each other in the same ratio.

Let $MPNQ$ be a great circle of the sphere, and $ABCD$ the circumscribed square. If the semicircle PMQ and the half square $PADQ$ are at the same time made to revolve about the diameter PQ , the semicircle will generate the sphere, while the half square will generate the cylinder circumscribed about that sphere.



The altitude AD of that cylinder is equal to the diameter PQ , the base of the cylinder is equal to the great circle, its diameter AB being equal to MN : hence (T. I.), the convex surface of the cylinder is equal to the circumference of the great circle multiplied by its diameter. This measure (T. IX.) is the same as that of the surface of the sphere: hence, *the surface of the sphere is equal to the convex surface of the circumscribed cylinder.*

But the surface of the sphere is equal to four great circles, hence the convex surface of the cylinder is also equal to four great circles; and adding the two bases, each equal to a great circle, the total surface of the circumscribed cylinder will be equal to six great circles: hence the surface of the sphere is to the total surface of the circumscribed cylinder as 4

is to 6, or as 2 is to 3, which is the first branch of the proposition.

In the next place, since the base of the circumscribed cylinder is equal to a great circle, and its altitude to the diameter, the volume of the cylinder (T. II.) will be equal to a great circle multiplied by its diameter. But (T. XII.) the volume of the sphere is equal to four great circles multiplied by a third of the radius; in other terms, to one great circle multiplied by $\frac{4}{3}$ of the radius, or by $\frac{2}{3}$ of the diameter. Hence the sphere is to the circumscribed cylinder as 2 to 3, and consequently the volumes of these two bodies are as their surfaces.

EIGHTH BOOK.

SPHERICAL GEOMETRY.

DEFINITIONS.

I. *A spherical triangle* is a portion of the surface of a sphere, bounded by three arcs of great circles.

Those arcs, named *the sides of the triangle*, are always supposed to be each less than a semi-circumference; the angles, which their planes form with each other, are *the angles of the triangle*.

II. A spherical triangle takes the name of *right-angled*, *isosceles*, *equilateral*, in the same cases as a rectilineal triangle.

III. *A spherical polygon* is a portion of the surface of a sphere, terminated by several arcs of great circles.

IV. *A lune* is that portion of the surface of a sphere, which is included between two great semicircles meeting in a common diameter.

V. *A spherical wedge*, or *ungula*, is that portion of the solid sphere which is included between the same great semicircles, and has the lune for its base.

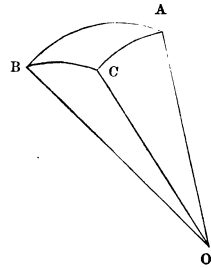
VI. *A spherical pyramid* is a portion of the solid sphere, included between the planes of a polyedral angle whose vertex is the centre; *the base* of the pyramid is the spherical polygon intercepted by the same planes.

VII. *The pole of a circle* is a point on the surface of the sphere equally distant from all the points of the circumference.

THEOREM I.

In every spherical triangle, any side is less than the sum of the other two.

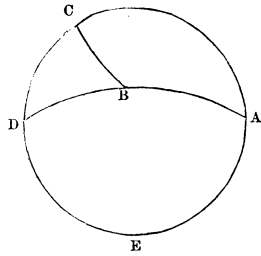
Let O be the centre of the sphere; and draw the radii OA, OB, OC. Imagine the planes AOB, AOC, COB; those planes will form a polyedral angle at the point O; and the angles AOB, AOC, COB will be measured by AB, AC, BC, the sides of the spherical triangle. But (B. V., T. XIX.) each of the three plane triangles composing a polyedral angle is less than the sum of the other two; hence, any side of the triangle ABC is less than the sum of the other two.



THEOREM II.

The sum of all the three sides of a spherical triangle is less than the circumference of a great circle.

Let ABC be any spherical triangle; produce the sides AB, AC till they meet again in D. The arcs ABD, ACD will be semi-circumferences—since (B. VII., T. VI., C. II.) two great circles always bisect each other. But in the triangle BCD we have (T. I.) the side $BC < BD + CD$: add $AB + AC$ to each; we shall have $AB + AC + BC < ABD + ACD$, that is to say, less than a circumference.

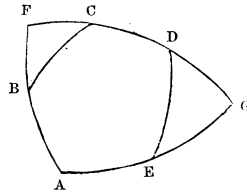


THEOREM III.

The sum of all the sides of any spherical polygon is less than the circumference of a great circle.

Let us take, for example, the pentagon ABCDE. Produce the sides AB, DC till they meet in F; then, since BC is less

than $BF + CF$, the perimeter of the pentagon $ABCDE$ will be less than that of the quadrilateral $AEDF$. Again, produce the sides AE, FD till they meet in G ; we shall have $ED < EG + DG$: hence the perimeter of the quadrilateral $AEDF$ is less than that of the triangle

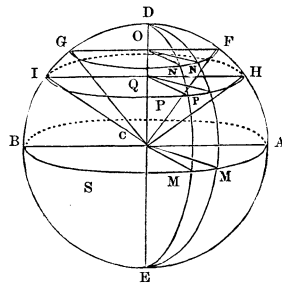


AFG , which last is itself less than the circumference of a great circle (T. II.): hence the perimeter of the polygon $ABCDE$ is less than this same circumference.

THEOREM IV.

If a diameter be drawn perpendicular to the plane of a great circle, its extremities will be the poles of that circle, and also of all small circles parallel to it.

For, DC being perpendicular to the plane AMB , is perpendicular to all the straight lines CA, CM, CB , etc., drawn through its foot in this plane; hence all the arcs DA, DM, DB , etc., are quarters of the circumference. So likewise are all the arcs EA, EM, EB , etc.; hence the points D and E are each equally distant from all the points of the circumference AMB ; therefore (D. VII.) they are the poles of that circumference.



Again, the radius DC , perpendicular to the plane AMB , is perpendicular to its parallel FNG ; hence (B. VII., T. VI., C. IV.) it passes through O , the centre of the circle FNG ; therefore, if the oblique lines DF, DN, DG be drawn, they will be equal (B. V., T. V.); but, the chords being equal, the arcs are equal: hence the point D is the pole of the small circle FNG ; and, for like reasons, the point E is the other pole.

Cor. I. Every arc DM drawn from a point in the arc of a great circle AMB to its pole, is a quarter of the circumference, which, for the sake of brevity, is usually named a *quadrant*; and this quadrant, at the same time, makes a right angle with the arc AM . For (B. V., T. XVI.), the line DC being perpen-

dicular to the plane AMC , every plane DMC passing through the line DC is perpendicular to the plane AMC ; hence the angles of these planes, or the angle AMD , is a right angle.

Cor. II. To find the pole of a given arc AM , draw the indefinite arc MD perpendicular to AM ; take MD equal to a quadrant: the point D will be one of the poles of the arc AM . Or thus: at the two points A and M , draw the arcs AD and MD perpendicular to AM ; their point of intersection, D , will be the pole required.

Cor. III. Conversely, if the distance of the point D from each of the points A and M be equal to a quadrant, the point D will be the pole of the arc AM ; and also the angles DAM , DMA will be right angles.

For, let C be the centre of the sphere, and draw the radii CA , CD , CM . Since the angles ACD , MCD are right, the line CD is perpendicular to the two straight lines CA , CM ; it is, therefore, perpendicular to their plane: hence the point D is the pole of the arc AM , and consequently the angles DAM , DMA are right.

Scholium. The properties of these poles enable us to describe arcs of a circle on the surface of a sphere with the same facility as on a plane surface. It is evident, for instance, that by turning the arc DF , or any other line extending to the same distance, round the point D , the extremity F will describe the small circle FNG ; and by turning the quadrant DFA round the point D , its extremity A will describe the arc of the great circle AM .

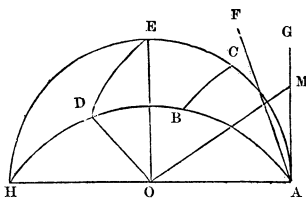
If the arc AM were required to be produced, and nothing were given but the points A and M through which it was to pass, we should first have to determine the pole D , by the intersection of two arcs described from the points A and M as centres, with a distance equal to a quadrant; the pole D being found, we might describe the arc AM and its prolongation from D as a centre, and with the same distance as before.

Lastly, if it be required from a given point P to draw a perpendicular on the given arc AM , produce this arc to S till the distance PS be equal to a quadrant; then from the pole S , and with the same distance, describe the arc PM , which will be the perpendicular required.

THEOREM V.

The angle formed by two arcs of great circles is equal to the angle formed by the tangents of these arcs at the point of intersection, and is therefore measured by the arc described from the point of intersection as a pole, between the sides, produced if necessary.

For the tangent AF drawn in the plane of the arc AB is perpendicular to the radius AO ; and the tangent AG drawn in the plane of the arc AC is perpendicular to the same radius AO : hence (B. V., .D. VI.) the angle FAG is equal to the angle contained by the planes OAB , OAC , which is that of the arcs AB , AC , and is named BAC .

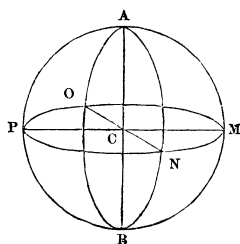


In like manner, if the arcs AD and AE are both quadrants, the lines OD , OE will be perpendicular to AO , and the angle DOE will still be equal to the angle of the planes AOD , AOE ; hence the arc DE is the measure of the angle contained by these planes, or of the angle CAB .

Cor. The angles of spherical triangles may be compared together, by means of the arcs of great circles described from their vertices as poles, and included between their sides; hence it is easy to make an angle of this kind equal to a given angle.

Scholium. Vertical angles, such as ACO and BCN , are equal; for either of them is still the angle formed by the two planes ACB , OCN .

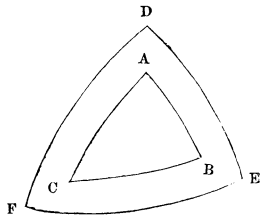
It is further evident, that, in the intersection of two arcs ACB , OCN , the two adjacent angles ACO , OCB taken together are equal to two right angles.



THEOREM VI.

If, with the vertices of a given triangle as poles, arcs be described, forming a new triangle, then will the vertices of this new triangle be the poles respectively of the sides of the given triangle.

For, the point A being the pole of the arc EF, the distance AE is a quadrant; the point C being the pole of the arc DE, the distance CE is likewise a quadrant: hence the point E is removed the distance of a quadrant from each of the points A and C; it is, therefore, the pole of the arc AC. It might be shown by the same method, that D is the pole of the arc BC, and F that of the arc AB.



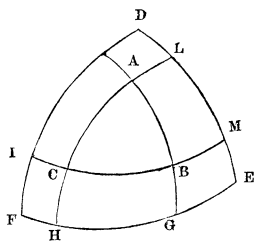
Cor. Hence the triangle ABC may be described by means of DEF, as DEF may by means of ABC.

THEOREM VII.

The same supposition being made as in the last Theorem, each angle in either one of the triangles will be measured by a semi-circumference MINUS the side lying opposite to it in the other triangle.

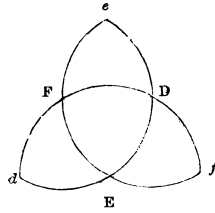
Produce the sides AB, AC, if necessary, till they meet EF in G and H. The point A being the pole of the arc GH, the angle A will be measured by that arc; but the arc EH is a quadrant, and likewise GF, E being the pole of AH, and F of AG: hence $EH + GF$ is equal to a semi-circumference. Now, $EH + GF$ is the same as $EF + GH$; hence the arc GH, which measures the angle A, is equal to a semi-circumference *minus* the side EF. In like manner, the angle B will be measured by $\frac{1}{2}$ circ. — DF; the angle C, by $\frac{1}{2}$ circ. — DE.

And this property must be reciprocal in the two triangles, since each of them is described in a similar manner by means



of the other. Thus we shall find the angles D, E, F of the triangle DEF to be measured respectively by $\frac{1}{2}$ circ. $-$ BC, $\frac{1}{2}$ circ. $-$ AC, $\frac{1}{2}$ circ. $-$ AB. The angle D, for example, is measured by the arc MI; but $MI + BC = MC + BI = \frac{1}{2}$ circ.; hence the arc MI, the measure of D, is equal to $\frac{1}{2}$ circ. $-$ BC; and so of the other angles.

Scholium. It must further be observed, that besides the triangle DEF, three others might be formed by the intersection of the three arcs DE, EF, DF. But the proposition immediately before us is applicable only to the central triangle, which is distinguished from the other three by the circumstance that the two angles A and D lie on the same side of BC, the two B and E on the same side of AC, and the two C and F on the same side of AB.

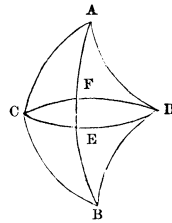


Various names have been given to the triangles ABC, DEF, but they are now more generally denominated *polar triangles*.

THEOREM VIII.

Any triangle on a sphere being given, another triangle may be constructed which shall have all its parts equal respectively to the corresponding parts of the given triangle.

Let ABC be the given triangle. With A as a pole, describe the arc CED passing through C; and with B as a pole, describe another arc CFD passing through C, and intersecting the former arc at D. Then will the triangle ADB have all its parts equal to the corresponding parts of the triangle ABC.



For, by construction, the side $AD = AC$, $BD = BC$, and AB is common; hence those two triangles have their sides equal each to each, and it is to be shown that the angles opposite these equal sides are also equal.

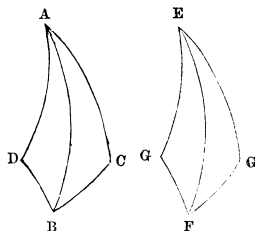
If the centre of the sphere is supposed to be at O, a polyedral angle may be conceived as formed at O by the three plane angles AOB, AOC, BOC; likewise another polyedral angle may be conceived as formed by the three plane angles AOB,

AOD, BOD. And because the sides of the triangle ABC are equal to those of the triangle ADB, the plane angles forming the one of these polyedral angles must be equal to the plane angles forming the other, each to each; but in this case the planes, in which the equal angles lie, are equally inclined to each other (B. V., T. XXI.); hence all the angles of the spherical triangle DAB are respectively equal to those of the triangle CAB, namely, $DAB = CAB$, $DBA = CBA$, and $ADB = ACB$; therefore the sides and the angles of the triangle ADB are equal to the sides and the angles of the triangle ACB. These triangles are called *symmetrical*.

THEOREM IX.

Two triangles on the same sphere, or on equal spheres, are equal in all their parts, when they have each an equal angle included between equal sides.

Suppose the side $AB = EF$, the side $AC = EG$, and the angle $BAC = FEG$; the triangle EFG may be placed on the triangle ABC, or on ABD symmetrical with ABC, just as two rectilineal triangles are placed upon each other when they have an equal angle included be-



tween equal sides. Hence all the parts of the triangle EFG will be equal to all the parts of the triangle ABC; that is, besides the three parts equal by hypothesis, we shall have the side $BC = FG$, the angle $ABC = EFG$, and the angle $ACB = EGF$.

THEOREM X.

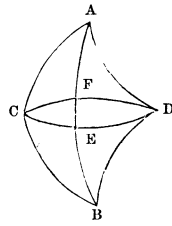
Two triangles on the same sphere, or on equal spheres, are equal in all their parts, when two angles and the included side of the one are equal to two angles and the included side of the other.

For one of those triangles, or the triangle symmetrical with it, may be placed on the other, and be made to coincide with it, as is obvious.

THEOREM XI.

If two triangles on the same sphere, or on equal spheres, have all their sides respectively equal, their angles will likewise be all respectively equal, the equal angles lying opposite the equal sides.

The truth is evident by T. VIII., where it was shown that, with three given sides, AB, AC, BC, there can only be two triangles ACB, ABD different as to the position of their parts, and equal as to the magnitude of those parts. Hence those two triangles, having all their sides respectively equal in both, must either be absolutely equal, or at least *symmetrically* so; in both of which cases, their corresponding angles must be equal, and lie opposite to equal sides.



THEOREM XII.

If two triangles on the same sphere, or on equal spheres, are mutually equiangular, they will also be mutually equilateral.

Let A and B be the two given triangles; P and Q their polar triangles. Since the angles are equal in the triangles A and B, the sides will be equal in the polar triangles P and Q (T. VIII.); but since the triangles P and Q are mutually equilateral, they must also (T. XI.) be mutually equiangular; and, lastly, the angles being equal in the triangles P and Q, it follows (T. VIII.) that the sides are equal in their polar triangles A and B. Hence the mutually equiangular triangles A and B are at the same time mutually equilateral.

Scholium. This proposition is not applicable to rectilinear triangles, in which equality among the angles indicates only proportionality among the sides. Nor is it difficult to account for the difference observable, in this respect, between spherical and rectilinear triangles. In the proposition now before us, as well as in the three last, which treat of the comparison of triangles, it is expressly required that the arcs be traced on the same sphere, or on equal spheres. Now, similar arcs are to each other as their

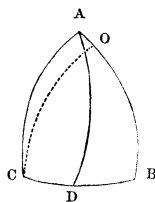
radii: hence, on equal spheres, two triangles cannot be similar without being equal; therefore it is not strange that equality among the angles should produce equality among the sides.

The case would be different if the triangles were drawn upon unequal spheres; there, the angles being equal, the triangles would be similar, and the homologous sides would be to each other as the radii of their spheres.

THEOREM XIII.

In every isosceles spherical triangle, the angles opposite the equal sides are equal; and, conversely, if two angles of a spherical triangle are equal, the triangle will be isosceles.

First. Suppose the side $AB = AC$, we shall have the angle $C = B$. For, if the arc AD be drawn from the vertex A to the middle point D of the base, the two triangles ABD , ACD will have all the sides of the one respectively equal to the corresponding sides of the other, namely, AD common, $BD = CD$, and $AB = AC$; hence, by the last proposition, their angles will be equal; therefore $B = C$.



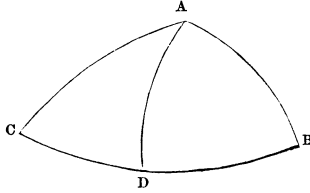
Secondly. Suppose the angle $B = C$; we shall have the side $AC = AB$. For, if not, let AB be the greater of the two; take $BO = AC$, and join OC . The two sides BO , BC are equal respectively to the two AC , BC ; the angle OBC contained by the first two is equal to ACB contained by the second two. Hence (T. IX.) the two triangles BOC , ACB have all their other parts equal; hence the angle $OCB = ABC$; but, by hypothesis, the angle $ABC = ACB$; hence we have $OCB = ACB$, which is absurd; therefore AB is not different from AC ; that is, the sides AB , AC , opposite to the equal angles B and C , are equal.

Scholium. The same demonstration proves that the angle $BAD = DAC$, and the angle $BDA = ADC$. Hence the two last are right angles; consequently, *the arc drawn from the vertex of an isosceles spherical triangle to the middle of the base, is at right angles to that base, and bisects the opposite angle.*

THEOREM XIV.

In any spherical triangle, the greater side is opposite the greater angle; and, conversely, the greater angle is opposite the greater side.

First. Suppose the angle $A > B$: make the angle $BAD = B$; then we shall have $AD = DB$ (T. XII.); but $AD + DC$ is greater than AC : hence, putting DB in place of AD , we shall have $DB + DC$, or $BC > AC$.

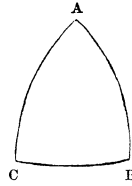


Secondly. If we suppose $BC > AC$, the angle BAC will be greater than ABC . For, if BAC were equal to ABC , we should have $BC = AC$; if BAC were less than ABC , we should then, as has just been shown, find $BC < AC$. Both these conclusions are false; hence the angle BAC is greater than ABC .

THEOREM XV.

The sum of all the angles in any spherical triangle is less than six right angles, and greater than two.

For, in the first place, every angle of a spherical triangle is less than two right angles (see the following Scholium); hence the sum of all the three is less than six right angles.



Secondly, the measure of each angle in the spherical triangle (T. VII.) is equal to the semi-circumference *minus* the corresponding side of the polar triangle; hence the sum of all the three is measured by three semi-circumferences *minus* the sum of all the sides of the polar triangle. Now (T. II.), this latter sum is less than a circumference; therefore, taking it away from three semi-circumferences, the remainder will be greater than one semi-circumference, which is the measure of two right angles. Hence, in the second place, the sum of all the angles in a spherical triangle is greater than two right angles.

Cor. I. The sum of all the angles in a spherical triangle is not constant, like that of all the angles in a rectilineal triangle; it

varies between two right angles and six, without ever arriving at either of these limits. Two given angles, therefore, do not serve to determine the third.

Cor. II. A spherical triangle may have two or even three angles right, two or three obtuse.

If the triangle ABC have two right angles B and C , the vertex A will (T. IV.) be the pole of the base BC , and the sides AB , AC will be quadrants.

If the angle A is also right, the triangle ABC will have all its angles right, and its sides quadrants. The tri-rectangular triangle is contained eight times in the surface of the sphere, as is evident from the figure in the next proposition, supposing the arc MN to be a quadrant.

Scholium. In all the preceding observations, we have supposed, in conformity with D. I., that our spherical triangles have always each of their sides less than a semi-circumference; from which it follows that any one of their angles is always less than two right angles. For (see the figure of T. II.), if the side AB is less than a semi-circumference, and AC is so likewise, both those arcs will require to be produced before they can meet in D . Now, the two angles ABC , CBD taken together, are equal to two right angles; hence the angle ABC itself is less than two right angles.

We may observe, however, that some spherical triangles do exist, in which certain of the sides are greater than a semi-circumference, and certain of the angles greater than two right angles. Thus, if the side AC is produced so as to form a whole circumference $ACDE$, the part which remains after subtracting the triangle ABC from the hemisphere, is a new triangle also designated by ABC , and having AB , BC , $AEDC$ for its sides. Here, it is plain, the side $AEDC$ is greater than the semi-circumference AED ; and, at the same time, the angle B opposite to it exceeds two right angles by the quantity CBD .

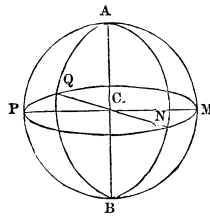
The triangles, whose sides and angles are so large, have been excluded from our definition; but the only reason was, that the solution of them, or the determination of their parts, is always reducible to the solution of such triangles as are comprehended by the definition. Indeed, it is evident enough, that if the sides and angles of the triangle ABC are known, it will be easy to

discover the angles and sides of the triangle which bears the same name, and is the difference between a hemisphere and the former triangle.

THEOREM XVI.

The surface of a lune is to the entire surface of the sphere, as the angle of the lune is to four right angles, or as the arc which measures the angle of the lune is to the circumference.

Suppose, in the first place, the arc MN to be to the circumference MNPQ as some one rational number is to another, as 5 to 48 for example. The circumference MNPQ being divided into 48 equal parts, MN will contain 5 of them; and if the pole A were joined with the several points of division, by as many quadrants, we should in the hemisphere AMNPQ have 48 triangles, all equal, because having all their parts equal. Hence the whole sphere must contain 96 of those partial triangles, and the lune AMBNA will contain 10 of them; hence the lune is to the sphere as 10 is to 96, or as 5 to 48; in other words, as the arc MN is to the circumference.



If the arc MN is not commensurable with the circumference, we may still show, by the mode of reasoning employed in the case of incommensurable magnitudes, that in this instance, also, the lune is to the sphere as MN is to the circumference.

Cor. I. Two lunes are to each other as their respective angles.

Cor. II. It was shown (T. XV., C. II.) that the whole surface of the sphere is equal to eight tri-rectangular triangles; hence, if the area of one such triangle is taken for unity, the surface of the sphere will be represented by 8. This granted, the surface of the lune whose angle is A will be expressed by 2 A (the angle A being always estimated from the right angle assumed as unity), since $2A : A :: 8 : 4$. Thus we have here two different unities: one for angles, being the right angle; the other for surfaces, being the tri-rectangular spherical triangle, or the triangle whose angles are all right, and whose sides are quadrants.

Scholium. The spherical ungula bounded by the planes AMB,

ANB, is to the whole sphere, as the angle A is to four right angles; for, the lunes being equal, the spherical unguulas will also be equal; hence two spherical unguulas are to each other as the angles formed by the planes which bound them.

THEOREM XVII.

Two symmetrical spherical triangles are equal in surface.

Let ABC, DEF be two symmetrical triangles; that is to say, two triangles having their sides

$$AB = DE,$$

$$AC = DF,$$

$$BC = EF,$$

and yet incapable of coinciding with each other: we are to show

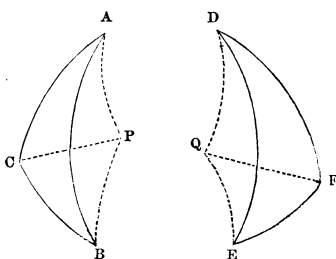
that the surface ABC is equal to the surface DEF.

Let P be the pole of the little circle passing through the three points A, B, C; from this point, draw (T. IV., S.) the equal arcs PA, PB, PC; at the point F, make the angle DFQ = ACP, the arc FQ = CP; and join DQ, EQ.

The sides DF, FQ are equal to the sides AC, CP; the angle DFQ = ACP; hence (T. IX.) the two triangles DFQ, ACP are equal in all their parts; hence the side DQ = AP, and the angle DQF = APC.

In the proposed triangles DEF, ABC, the angles DFE, ACB opposite to the equal sides DE, AB being equal (T. VIII.), if the angles DFQ, ACP, which are equal by construction, be taken away from them, there will remain the angle QFE equal to PCB. Also the sides QF, FE are equal to the sides PC, CB; hence the two triangles FQE, CPB are equal in all their parts: hence the side QE = PB, and the angle FQE = CPB.

Now, observing that the triangles DFQ, ACP, which have their sides respectively equal, are at the same time isosceles, we shall see them to be capable of mutual adaptation, when applied to each other; for, having placed PA on its equal QF, the side PC will fall on its equal QD, and thus the two triangles will exactly coincide; hence they are equal, and the surface DQF =



APC. For a like reason, the surface $FQE = CPB$, and the surface $DQE = APB$; hence we have

$$\begin{aligned} DQF + FQE - DQE &= APC + CPB - APB, \\ \text{that is,} \quad DEF &= ABC; \end{aligned}$$

therefore, the two symmetrical triangles ABC , DEF are equal in surface.

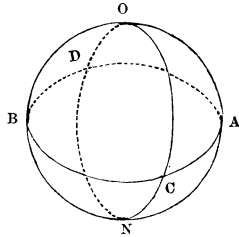
Scholium. The poles P and Q might lie within the triangles ABC , DEF ; in which case, it would be requisite to add the three triangles DQF , FQE , DQE together, in order to make up the triangle DEF ; and, in like manner, to add the three triangles APC , CPB , APB together, in order to make up the triangle ABC . In all other respects, the demonstration and the result would still be the same.

THEOREM XVIII.

If two great circles intersect each other on the surface of a hemisphere, the sum of the opposite triangles thus formed will be equivalent to the lune whose angle is equal to the angle formed by the circles.

Let the circumference AOB , COD intersect on the hemisphere $OACBD$; then will the opposite triangles AOC , BOD be equal to the lune whose angle is BOD .

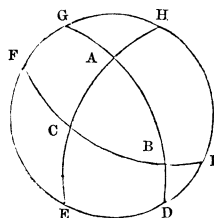
For, producing the arcs OB , OD in the other hemisphere, till they meet in N , the arc OBN will be a semi-circumference, and AOB one also; and taking OB from each, we shall have $BN = AO$. For a like reason, we have $DN = CO$, and $BD = AC$. Hence the two triangles AOC , BDN have their three sides respectively equal; besides, they are so placed as to be symmetrical; hence (T. XVII.) they are equal in surface, and the sum of the triangles AOC , BOD is equal to the lune $OBND$. whose angle is BOD .



THEOREM XIX.

The surface of any spherical triangle is measured by the excess of the sum of its three angles above two right angles.

Let ABC be the proposed triangle: produce its sides till they meet the great circle $DEFGHI$, drawn anywhere without the triangle. By the last proposition, the two triangles ADE , AGH are together equal to the lune whose angle is A , and which is measured (T. XVI., C. II.) by $2A$; hence we have $ADE + AGH = 2A$; and, for a like reason, $BGF + BID = 2B$, and $CIH + CFE = 2C$. But the sum of those six triangles exceeds the hemisphere by twice the triangle ABC , and the hemisphere is represented by 4 ; therefore twice the triangle ABC is equal to $2A + 2B + 2C - 4$, and consequently once $ABC = A + B + C - 2$; hence, every spherical triangle is measured by the sum of all its angles *minus* two right angles.



Cor. I. However many right angles there be contained in this measure, just so many tri-rectangular triangles, or eighths of the sphere, which (T. XVI., C. II.) are the unit of surface, will the proposed triangle contain. If the angles, for example, are each equal to $\frac{1}{3}$ of a right angle, the three angles will amount to 4 right angles, and the proposed triangle will be represented by $4 - 2$, or 2; therefore it will be equal to two tri-rectangular triangles, or to the fourth part of the whole surface of the sphere.

Cor. II. The spherical triangle ABC is equal to the lune whose angle is $\frac{A + B + C}{2} - 1$. Likewise the spherical pyramid which has ABC for its base, is equal to the spherical ungula whose angle is $\frac{A + B + C}{2} - 1$.

Scholium. While the spherical triangle ABC is compared with the tri-rectangular triangle, the spherical pyramid, which has ABC for its base, is compared with the tri-rectangular pyramid, and the same ratio is found to subsist between them. The polyedral angle at the vertex of the pyramid is, in like manner, compared with the polyedral angle at the vertex of the tri-rect-

angular pyramid. These comparisons are founded on the coincidence of the corresponding parts. If the bases of the pyramids coincide, the pyramids themselves will evidently coincide, and likewise the polyedral angles at their vertices. From this, the following consequences are deduced :

First. Two triangular spherical pyramids are to each other as their bases ; and since a polygonal pyramid may always be divided into a certain number of triangular ones, it follows that any two spherical pyramids are to each other as the polygons which form their bases.

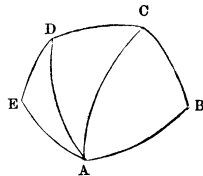
Secondly. The polyedral angles at the vertices of those pyramids are also as their bases ; hence, for comparing any two polyedral angles, we have merely to place their vertices at the centres of two equal spheres, and the polyedral angles will be to each other as the spherical polygons intercepted between their planes or faces.

The vertical angle of the tri-rectangular pyramid is formed by three planes at right angles to each other : this angle, which may be called a *right polyedral angle*, will serve as a very natural unit of measure for all other polyedral angles ; and if so, the same number that exhibits the area of a spherical polygon, will exhibit the measure of the corresponding polyedral angle. If the area of the polygon is $\frac{3}{4}$, for example ; in other words, if the polygon is $\frac{3}{4}$ of the tri-rectangular polygon, then the corresponding polyedral angle will also be $\frac{3}{4}$ of the right polyedral angle.

THEOREM XX.

The surface of a spherical polygon is measured by the sum of all its angles, MINUS the product of two right angles by the number of sides in the polygon MINUS two.

From one of the vertices A, let diagonals AC, AD be drawn to all the other vertices ; the polygon ABCDE will be divided into as many triangles, *minus* two, as it has sides. But the surface of each triangle is measured by the sum of all its angles *minus* two right angles, and the sum of the angles in all the triangles is evidently the same as that of all the angles in the polygon ; hence the sur-



face of the polygon is equal to the sum of all its angles, diminished by twice as many right angles as it has sides *minus* two.

Scholium. Let s be the sum of all the angles in a spherical polygon, and n the number of its sides; the right angle being taken for unity, the surface of the polygon will be measured by $s - 2(n - 2)$, or $s - 2n + 4$.

REGULAR POLYEDRONS.

Since the faces of any regular polyedron are all equal and regular polygons (B. VI., D. XII.), it follows that all its polyedral angles must be equal.

There can be only five kinds of regular polyedrons.

For, we already know that *three* faces at least are necessary to form a polyedral angle; and moreover that the sum of all the plane angles which form the polyedral angle is less than 4 right angles. It is sufficient then to consider all the cases in which the angles of equal and regular polygons, taken 3 at a time, 4 at a time, 5 at a time, etc., do not equal or exceed 4 right angles. Now,

1. The angle of an equilateral triangle being $\frac{2}{3}$ of a right angle, we may construct a polyedral angle by using *three, four, or five* equilateral triangles for faces. But we cannot use six or a greater number, since $\frac{2}{3}$ of a right angle $\times 6 = 4$ right angles.

2. The angle of a square being 1 right angle, we may form a polyedral angle by using *three* squares for the faces. But we could not use four or a greater number, since 1 right angle $\times 4 = 4$ right angles.

3. The angle of a regular pentagon being $\frac{3}{5}$, a polyedral angle may be formed with *three* regular pentagons, which gives $\frac{3}{5} \times 3 = 3\frac{3}{5}$.

Since the angle of a regular hexagon is $\frac{2}{3}$, and $\frac{2}{3} \times 3 = 4$, it follows, that a polyedral angle cannot be formed with regular hexagons for faces. And of course, regular polygons of a greater number of sides cannot be used in forming a polyedral angle.

Thus, the only possible regular polyedrons are those of which each polyedral angle is formed by *three, four, or five* equilateral

triangles, by *three* squares, and by *three* regular pentagons, making in all *five* polyedrons.

When *three* equilateral triangles are used in forming each polyedral angle, the whole number of faces of the regular polyedron will be *four*, and it is called a *Tetraedron*.

When *four* equilateral triangles unite in forming each polyedral angle, the polyedron will have *eight* faces, and it is called an *Octaedron*.

When *five* equilateral triangles are used in forming a polyedral angle, the polyedron will have *twenty* faces, and it is then called an *Icosaedron*.

When *three* squares are used in forming each polyedral angle of a regular polyedron, there will be *six* faces, and the polyedron is called a *Hexagon*, or *Cube* (B. VI., D. V.).

When *three* regular pentagons are used for each polyedral angle, the number of faces will be *twelve*, and the solid is called a *Dodecaedron*.

These five regular solids may obviously be inscribed in a sphere, as well as circumscribed about a sphere; or a sphere may be inscribed in each, and circumscribed about each. The inscribed and circumscribed spheres have a common centre, which may be regarded as the centre of the polyedron.

If we suppose planes to pass through the centre of a regular polyedron and each of its edges, they will divide the polyedron into as many equal pyramids as the polyedron has faces, since each face of the polyedron will thus become a base of a pyramid, whose altitude will be the radius of the inscribed sphere; and since each pyramid is measured by the area of its base multiplied by one third of its altitude, it follows that the volume of any regular polyedron has for its measure its surface multiplied by one third the radius of its inscribed sphere.

Were we to regard the sphere as a regular polyedron of an infinite number of infinitely small and equal faces, we should, from the above, at once infer that the volume of a sphere has for its measure its surface multiplied by one third its radius.

Whether we regard the sphere as a regular polygon or not, it is essentially a *regular body*.

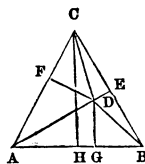
Hence, there are then six *regular bodies*: the *Tetraedron*, *Hexaedron*, *Octaedron*, *Dodecaedron*, *Icosaedron*, and the *Sphere*.

APPLICATION OF ALGEBRA TO THE SOLUTION OF GEOMETRICAL PROBLEMS.

PROBLEM I.

In an equilateral triangle, having given the lengths of the three perpendiculars drawn from a certain point within it to the three sides, to determine its side.

Let ABC be the equilateral triangle, and DE, DF, DG the perpendiculars from the point D upon the sides respectively. Denote these perpendiculars by a, b, c , in order, and the side of the triangle ABC by $2x$. Then, if the perpendicular CH be drawn,



$$CH = \sqrt{AC^2 - AH^2} = \sqrt{4x^2 - x^2} = x\sqrt{3}.$$

The area of the triangle $ADB = \frac{1}{2} AB \cdot DG = cx$. Similarly the triangle $BDC = ax$, the triangle $CDA = bx$, and the triangle $ACB = \frac{1}{2} AB \cdot CH = x^2\sqrt{3}$. Also, $BDC + CDA + ADB = ABC$; that is, in symbols,

$$x^2\sqrt{3} = (a + b + c)x, \text{ and } x = \frac{a + b + c}{\sqrt{3}},$$

which is half the side of the triangle sought.

Cor. From the resulting equation, we have

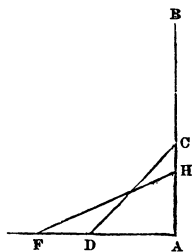
$$x\sqrt{3} = a + b + c:$$

we also had $CH = x\sqrt{3}$. Hence $CH = a + b + c$; or the whole perpendicular CH is equal to the sum of the three smaller perpendiculars from D upon the sides, whenever the point D is taken *within* the triangle. Had the point D been taken *without* the triangle, the perpendicular upon the side which subtends the angle within which the point lies would become negative. Thus, had the point been without the triangle, but between the sides AB, AC *produced*, then $CH = DF + DG - DE$.

PROBLEM II.

A May-pole was broken off by the wind, and its top struck the ground twenty feet from the base; and, being repaired, was broken a second time five feet lower, and its top struck the ground ten feet farther from the base. What was the height of the May-pole?

Let AB be the unbroken May-pole, C and H the points in which it was successively broken, and D and F the corresponding points at which the top B struck the ground. Then will CAD and HAF be right-angled triangles.



Put $BC = CD = x$, $CA = y$, $AD = a$, $AF = b$, and $CH = c$. Then $AB = x + y$, $BH = HF = x + c$, and $HA = y - c$; therefore (B. III., T. XIV.), we have

$$y^2 + a^2 = x^2, \quad (1)$$

$$(y - c)^2 + b^2 = (x + c)^2. \quad (2)$$

Expanding (2) and subtracting (1) from it, we have, after a slight reduction,

$$x + y = \frac{b^2 - a^2}{2c} = 50 \text{ feet, the required height.}$$

PROBLEM III.

A statue eighty feet high stands on a pedestal fifty feet high, and to a spectator on the horizontal plane, they subtend equal angles; required the distance of the observer from the base, the height of the eye being five feet.

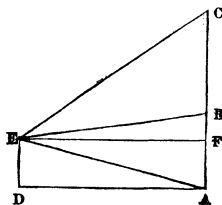
Let $AB = a$, the height of the pedestal;

$BC = b$, the height of the statue;

$DE = c$, the height of the eye from ground,

and $DA = EF = x$, the distance sought.

Then



$$EC^2 = EF^2 + CF^2 = x^2 + (a + b - c)^2,$$

and

$$EA^2 = EF^2 + ED^2 = x^2 + c^2.$$

But, since the angle $CEB = BEA$, we have (B. III., T. XII.)

$$EC : EA :: CB : BA, \text{ or,}$$

$$EC^2 : EA^2 :: CB^2 : BA^2;$$

or, in symbols, $x^2 + (a + b - c)^2 : x^2 + c^2 :: b^2 : a^2$.

From this proportion, we readily deduce

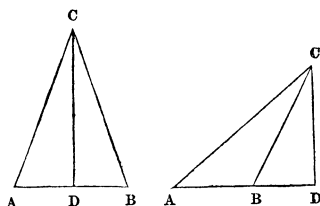
$$x = \pm \sqrt{\left(\frac{a(a-c)^2 + b(a^2 - c^2)}{b-a} \right)};$$

the double sign merely indicating that this value of x may be measured either way, from A towards D, or from D towards A.

PROBLEM IV.

To determine the area of a triangle when the three sides are given.

Let ABC be the triangle. From the angle C draw the perpendicular CD, which in the second figure falls without the triangle, and meets the base AB produced.



Denote the sides respectively opposite the angles A, B, C, by a , b , c , and the perpendicular CD by p . The right-angled triangles CDA, CDB give

$$b^2 = DA^2 + p^2; \quad a^2 = DB^2 + p^2; \text{ consequently,}$$

$$b^2 - a^2 = DA^2 - DB^2 = (DA + DB)(DA - DB).$$

Hence, in the first figure,

$$DA - DB = \frac{b^2 - a^2}{c},$$

and in the second figure,

$$DA + DB = \frac{b^2 - a^2}{c}.$$

Now, since half the difference of two quantities added to half their sum gives the greater, we have, in both cases,

$$\text{the greater segment } DA = \frac{b^2 - a^2}{2c} + \frac{c}{2} = \frac{b^2 + c^2 - a^2}{2c}.$$

$$\text{Hence, } p = \sqrt{b^2 - DA^2} = \frac{\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}}{2c}.$$

Since the difference of two squares is equal to the product of the sum of the two roots into their difference, we have

$$\begin{aligned} p &= \frac{\sqrt{(b^2 + 2bc + c^2 - a^2)(a^2 - b^2 + 2bc - c^2)}}{2c} \\ &= \frac{\sqrt{[(b+c)^2 - a^2] \times [a^2 - (b-c)^2]}}{2c} \\ &= \frac{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}{2c}. \quad (1) \end{aligned}$$

Multiplying this perpendicular by half the base, we have for the area,

$$\begin{aligned} &\frac{1}{4} \sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}, \text{ or} \\ &\left[\left(\frac{a+b+c}{2} \right) \left(\frac{-a+b+c}{2} \right) \left(\frac{a-b+c}{2} \right) \left(\frac{a+b-c}{2} \right) \right]^{\frac{1}{2}}. \quad (2) \end{aligned}$$

Hence we may find the area of a triangle, when the three sides are known, by this

RULE.—Take half the sum of the three sides, and from this half sum subtract each side separately; then take the square root of the continued product of the half sum and the three remainders, and it will be the area.

PROBLEM V.

Given the three sides a, b, c of a triangle, to find:

1. *The three perpendiculars from the angles upon the opposite sides;*
2. *The area of the triangle;*
3. *The radius of the circumscribed circle;*
4. *The radius of the inscribed circle;*
5. *The radii of the escribed circles.*

Let ABC be the triangle; and let a, b, c denote the sides opposite the angles A, B, C respectively, and P_1, P_2, P_3 the perpendiculars drawn from the angles A, B, C; Δ the area of the triangle; R the radius of the circumscribing circle; r that of the inscribed circle; and r_1, r_2, r_3 the radii of the three escribed circles, which touch the sides a, b, c externally.

An *escribed* circle has already been defined as a circle which

touches one of the sides of a triangle exteriorly, and the other two sides produced. (See B. II., T. XII., S. II.)

We have already found (Prob. IV.) the perpendiculars to be

$$P_1 = \frac{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}{2a}, \quad (1)$$

$$P_2 = \frac{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}{2b}, \quad (2)$$

$$P_3 = \frac{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}{2c}. \quad (3)$$

We have, also, under the same Problem, found the area to be

$$\Delta = \left\{ \left(\frac{a+b+c}{2} \right) \left(\frac{-a+b+c}{2} \right) \left(\frac{a-b+c}{2} \right) \left(\frac{a+b-c}{2} \right) \right\}^{\frac{1}{2}}. \quad (4)$$

We have (B. III., T. XXV.) $\Delta = \frac{abc}{4R}$, we also have $\Delta = \frac{P_1 a}{2}$;

therefore,
$$\frac{abc}{4R} = \frac{P_1 a}{2}.$$

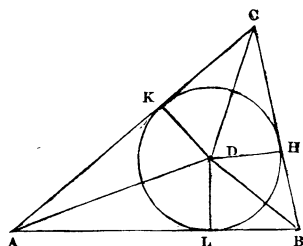
Hence,
$$R = \frac{bc}{2P_1}.$$

Substituting for P_1 its value already found, we have

$$R = \frac{abc}{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}. \quad (5)$$

The sum of the areas of the three triangles ADB, BDC, CDA equals the area ABC. But

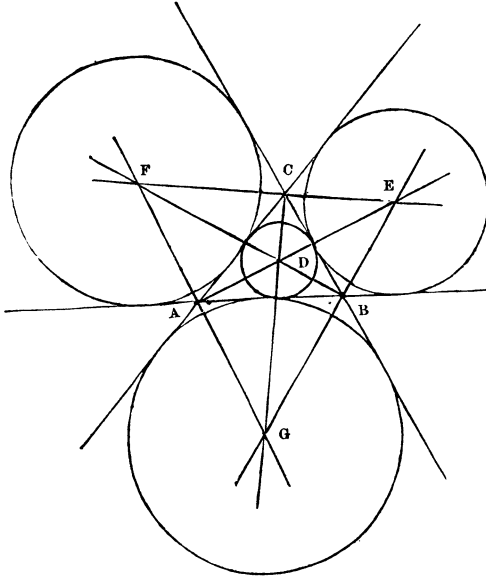
$$\begin{aligned} \text{ADB} &= \frac{1}{2}rc, \\ \text{BDC} &= \frac{1}{2}ra, \\ \text{CDA} &= \frac{1}{2}rb; \end{aligned}$$



hence,
$$\frac{a+b+c}{2} \times r = \Delta, \text{ and} \quad (6)$$

$$r = \frac{2\Delta}{a+b+c} = \left\{ \frac{(-a+b+c)(a-b+c)(a+b-c)}{4(a+b+c)} \right\}^{\frac{1}{2}}. \quad (7)$$

We will now seek the radius r_1 of the escribed circle, whose centre is at E.



Area $EBA + ECA - EBC$ is evidently equal to the area ABC . Area EBA is equal to the base AB multiplied by half the perpendicular drawn from E upon AB produced; hence area $EBA = c \times \frac{1}{2} r_1 = \frac{1}{2} cr_1$. In a similar way, we find area $ECA = \frac{1}{2} br_1$, and area $EBC = \frac{1}{2} ar_1$.

Therefore we have $\frac{b+c-a}{2} \times r_1 = \Delta$, and (8)

$$r_1 = \frac{2\Delta}{b+c-a} = \left\{ \frac{(a+b+c)(a-b+c)(a+b-c)}{4(-a+b+c)} \right\}^{\frac{1}{2}}. \quad (9)$$

By simply permuting, we obtain

$$r_2 = \left\{ \frac{a+b+c}{4(a-b+c)} \frac{(-a+b+c)(a+b-c)}{4} \right\}^{\frac{1}{2}}, \quad (10)$$

$$r_3 = \left\{ \frac{a+b+c}{4(a+b-c)} \frac{(-a+b+c)(a-b+c)}{4} \right\}^{\frac{1}{2}}. \quad (11)$$

Equation (8) readily gives

$$\frac{1}{r_1} = \frac{-a+b+c}{2\Delta};$$

and, in a similar manner,

$$\frac{1}{r_2} = \frac{a-b+c}{2\Delta}, \text{ and } \frac{1}{r_3} = \frac{a+b-c}{2\Delta};$$

therefore,
$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{a+b+c}{2\Delta}.$$

But we have already from equation (7),

$$r = \frac{2\Delta}{a+b+c}, \text{ or } \frac{1}{r} = \frac{a+b+c}{2\Delta};$$

therefore, we have
$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}. \quad (12)$$

Since any side of a triangle, multiplied by the perpendicular which meets it from the opposite angle, gives double the area of the triangle, we have $2\Delta = aP_1$, or

$$\frac{2\Delta}{P_1} = a. \quad (13)$$

In a similar manner, we have

$$\frac{2\Delta}{P_2} = b, \quad (14)$$

$$\frac{2\Delta}{P_3} = c. \quad (15)$$

Taking the product of (13), (14), and (15), we have

$$\frac{8\Delta^3}{P_1P_2P_3} = abc. \quad (16)$$

Again, we have $2RP_1 = bc, \quad (17)$

$$2RP_2 = ac, \quad (18)$$

$$2RP_3 = ab. \quad (19)$$

Taking the product of (17), (18), and (19), we have

$$8R^3P_1P_2P_3 = a^2b^2c^2. \quad (20)$$

Extracting the cube root of the product of (16) and (20), we find

$$4R\Delta = abc. \quad (21)$$

Dividing (20) by the square of (16), we find

$$\frac{R^3P_1^3P_2^3P_3^3}{8\Delta^6} = 1, \text{ or } RP_1P_2P_3 = 2\Delta^2. \quad (22)$$

By taking the continued product of (7), (9), (10), and (11), we have

$$r_1r_2r_3 = \frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{16} = \Delta^2. \quad (23)$$

By multiplying (5) and (7) together, or (7) and (21), we have

$$2Rr = \frac{abc}{a+b+c}. \quad (24)$$

By a similar multiplication, we find

$$\left. \begin{aligned} 2Rr_1 &= \frac{abc}{-a+b+c}, \\ 2Rr_2 &= \frac{abc}{a-b+c}, \\ 2Rr_3 &= \frac{abc}{a+b-c}. \end{aligned} \right\} \quad (25)$$

By combining the values of (9), (10), and (11), by two and two, we find

$$r_1r_2 + r_1r_3 + r_2r_3 = \left(\frac{a+b+c}{2} \right)^2. \quad (26)$$

We may also deduce

$$\frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}. \quad (27)$$

$$\frac{1}{2}Rr = \frac{\Delta^2}{P_1P_2 + P_2P_3 + P_3P_1}. \quad (28)$$

$$\left. \begin{aligned} P_1 &= \frac{2r_2r_3}{r_2 + r_3}, \\ P_2 &= \frac{2r_1r_3}{r_1 + r_3}, \\ P_3 &= \frac{2r_1r_2}{r_1 + r_2}. \end{aligned} \right\} \quad (29)$$

$$\left. \begin{aligned} \frac{1}{P_1} + \frac{1}{P_2} &= \frac{1}{P_3} + \frac{1}{r_3}, \\ \frac{1}{P_1} + \frac{1}{P_3} &= \frac{1}{P_2} + \frac{1}{r_2}, \\ \frac{1}{P_2} + \frac{1}{P_3} &= \frac{1}{P_1} + \frac{1}{r_1}. \end{aligned} \right\} \quad (30)$$

$$r_1 + r_2 + r_3 = 4R + r. \quad (31)$$

Any of the foregoing expressions, when properly translated into common language, leads to a theorem. We will translate some of the most interesting ones.

Equation (12) gives the following

THEOREM. *The reciprocal of the radius of the inscribed circle is equal to the sum of the reciprocals of the radii of the three escribed circles.*

Equation (21) yields the following

THEOREM. *Four times the radius of the circumscribed circle, into the area of the triangle, is equal to the continued product of the three sides.*

Equation (22) gives this

THEOREM. *The radius of the circumscribed circle, into the continued product of the three perpendiculars, is equal to twice the square of the area.*

Equation (23) gives this

THEOREM. *The radius of the inscribed circle, into the continued product of the radii of the three escribed circles, is equal to the square of the area.*

Equation (27) gives this

THEOREM. *The sum of the reciprocals of the three perpendiculars is equal to the sum of the reciprocals of the three radii of the escribed circles.*

By combining the equations already formed, new ones would arise, which might still afford interest. Thus, by a comparison of equations (22) and (23), we find

$$R P_1 P_2 P_3 = 2 r r_1 r_2 r_3; \quad (32)$$

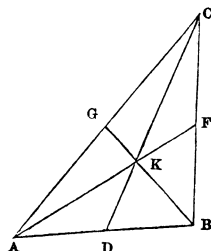
which gives this

THEOREM. *The radius of the circumscribed circle, into the continued product of the three perpendiculars, is equal to the diameter of the inscribed circle, into the continued product of the three radii of the escribed circles.*

PROBLEM VI.

Given the three sides of a triangle, to find the three lines drawn from the angles to the middle points of the opposite sides.

Let ABC be the triangle. Denote the sides opposite the angles A, B, C, by a , b , c respectively; also denote the lines drawn from the angles A, B, C to the middle points of the opposite sides, by m_1 , m_2 , m_3 respectively. Then (B. III., T. XVII.), we shall have



$$2 AF^2 + 2 BF^2 = AB^2 + AC^2, \text{ or}$$

$$2 m_1^2 + \frac{1}{2} a^2 = c^2 + b^2;$$

$$\text{which gives} \quad 4 m_1^2 = -a^2 + 2b^2 + 2c^2. \quad (1)$$

In a similar manner, we find

$$4 m_2^2 = -b^2 + 2c^2 + 2a^2, \quad (2)$$

$$4 m_3^2 = -c^2 + 2a^2 + 2b^2. \quad (3)$$

Equations (1), (2), and (3), readily give

$$\left. \begin{aligned} m_1 &= \frac{1}{2} \sqrt{-a^2 + 2b^2 + 2c^2}, \\ m_2 &= \frac{1}{2} \sqrt{-b^2 + 2c^2 + 2a^2}, \\ m_3 &= \frac{1}{2} \sqrt{-c^2 + 2a^2 + 2b^2}, \end{aligned} \right\} \quad (4)$$

We will now deduce a few remarkable relations, which, when properly translated, will give some beautiful theorems.

Taking the sum of (1), (2), and (3), we obtain

$$4 (m_1^2 + m_2^2 + m_3^2) = 3 (a^2 + b^2 + c^2). \quad (5)$$

If we take the product of (1) and (2), we shall have

$$16 m_1^2 m_2^2 = -2a^4 + 5a^2 b^2 + 2a^2 c^2 - 2b^4 + 2b^2 c^2 + 4c^4. \quad (6)$$

By simply permuting, we find

$$16 m_2^2 m_3^2 = -2b^4 + 5b^2 c^2 + 2b^2 a^2 - 2c^4 + 2c^2 a^2 + 4a^4, \quad (7)$$

$$16 m_3^2 m_1^2 = -2c^4 + 5c^2 a^2 + 2c^2 b^2 - 2a^4 + 2a^2 b^2 + 4b^4. \quad (8)$$

Taking the sum of (6), (7), and (8), we obtain

$$16 (m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2) = 9 (a^2 b^2 + b^2 c^2 + c^2 a^2). \quad (9)$$

If, from the square of (5), we subtract twice (9), we shall have

$$16 (m_1^4 + m_2^4 + m_3^4) = 9 (a^4 + b^4 + c^4). \quad (10)$$

The point where these lines trisect each other, is the centre of gravity of the triangle. If we denote the distances AK, BK, CK, by d_1, d_2, d_3 respectively, we shall have $m_1 = \frac{2}{3} d_1, m_2 = \frac{2}{3} d_2, m_3 = \frac{2}{3} d_3$ (B. I., T. XXXVIII). These values substituted in (5), (9), and (10), cause them to become

$$3 (d_1^2 + d_2^2 + d_3^2) = a^2 + b^2 + c^2, \quad (11)$$

$$9 (d_1^2 d_2^2 + d_2^2 d_3^2 + d_3^2 d_1^2) = a^2 b^2 + b^2 c^2 + c^2 a^2, \quad (12)$$

$$9 (d_1^4 + d_2^4 + d_3^4) = a^4 + b^4 + c^4. \quad (13)$$

Equation (5) is equivalent to the following

THEOREM. *Four times the sum of the squares of the lines drawn from the angles of a triangle to the middle points of the opposite sides, is equal to three times the sum of the squares of the sides.*

Equation (9) gives this

THEOREM. *Sixteen times the sum of the products, taken two at a time, of the squares of the lines drawn from the angles of a triangle to the middle points of the opposite sides, is equal to nine times the sum of the products, taken two at a time, of the squares of the sides.*

Equation (10) gives this

THEOREM. *Sixteen times the sum of the fourth powers of the lines drawn from the angles of a triangle to the middle points of the opposite sides, is equal to nine times the sum of the fourth powers of the sides.*

Equations (11), (12), and (13) would lead to beautiful theorems in reference to the distances of the centre of gravity of three equal bodies, and the mutual distances of the bodies themselves. For it is evident that, instead of considering the triangle as material, we may suppose equal weights placed at its vertices; and, under this point of view, equation (11) will give the following

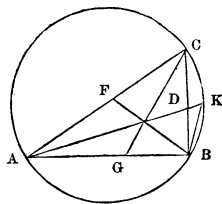
THEOREM. *Three times the sum of the squares of the distances of three equal bodies from their common centre of gravity, is equal to the sum of the squares of their mutual distances.*

PROBLEM VII.

Given the three sides of a triangle, to find the lines bisecting the angles, and terminating in the opposite sides.

Let the sides opposite the angles A , B , C be denoted by a , b , c respectively; also denote the lines which bisect the angles A , B , C respectively by l_1 , l_2 , l_3 .

Pass a circumference through the three points A , B , C , and produce l_1 to meet it at K ; join BK .



The two triangles ABK, ADC are similar, and give

$$AB : AD :: AK : AC;$$

consequently we have

$$AB \times AC = AD \times AK = AD \times (AD + DK) = AD^2 + AD \times DK;$$

which finally becomes

$$AB \times AC = AD^2 + BD \times DC,$$

since $BD \times DC = AD \times DK$ (B. IV., T. III., S. II.). In symbols this becomes

$$bc = l_1^2 + BD \times DC. \quad (1)$$

Again, we have (B. III., T. XII.)

$$AB : AC :: BD : DC.$$

$$\text{Consequently, } \begin{aligned} AB + AC : AB :: BD + DC : BD, \\ AB + AC : AC :: BD + DC : DC; \end{aligned}$$

$$\text{or, symbols, } \begin{aligned} c + b : c :: a : BD, \\ c + b : b :: a : DC. \end{aligned}$$

$$\text{Hence } BD = \frac{ac}{c+b}; \quad DC = \frac{ab}{c+b}.$$

These values of BD, DC, cause (1) to become

$$cb = l_1^2 + \frac{a^2bc}{(c+b)^2}.$$

This readily gives $l_1^2 = cb - \frac{a^2bc}{(c+b)^2}$, or

$$l_1^2 = \frac{cb(a+b+c)(-a+b+c)}{(c+b)^2}. \quad (2)$$

In a similar manner, we find

$$l_2^2 = \frac{ac(a+b+c)(a-b+c)}{(a+c)^2}, \quad (3)$$

$$l_3^2 = \frac{ba(a+b+c)(a+b-c)}{(b+a)^2}. \quad (4)$$

Equations (2), (3), (4), give

$$\left. \begin{aligned} l_1 &= \frac{\sqrt{cb(a+b+c)(-a+b+c)}}{c+b}, \\ l_2 &= \frac{\sqrt{ac(a+b+c)(a-b+c)}}{a+c}, \\ l_3 &= \frac{\sqrt{ba(a+b+c)(a+b-c)}}{b+a}. \end{aligned} \right\} \quad (5)$$

Taking the continued product of these values given by (5), we find

$$l_1 l_2 l_3 = \frac{abc(a+b+c) \sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}{(a+b)(b+c)(c+a)}.$$

The expression

$$\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)},$$

is equal to four times the area of the triangle (see P. IV.), which we will denote by 4Δ ; so that we shall have

$$l_1 l_2 l_3 = \frac{4abc(a+b+c)\Delta}{(a+b)(b+c)(c+a)}. \quad (6)$$

PROBLEM VIII.

To determine a right-angled triangle, having given the hypotenuse and difference of two lines drawn from the two acute angles to the centre of the inscribed circle.

Let $AC = h$; $AO = x + d$; $CO = x - d$; d being half the difference of the lines AO and CO . Produce AO , and draw CD perpendicular to AO thus produced. Then, since the angle COD is equal to the sum of CAO and ACO , and since CAO is half the angle CAB , and ACO the half of ACB , it follows that COD is half a right angle; consequently OCD is also half a right angle: therefore OD is to OC as the side of a square is to its diagonal; that is, as 1 to $\sqrt{2}$. Hence

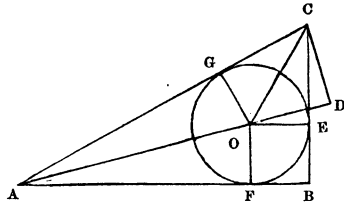
$$OD = CD = \frac{x-d}{\sqrt{2}}, \text{ and } AD = x+d + \frac{x-d}{\sqrt{2}}.$$

Again, $AC^2 = AD^2 + CD^2$; or which is in symbols

$$h^2 = \left(x+d + \frac{x-d}{\sqrt{2}}\right)^2 + \left(\frac{x-d}{\sqrt{2}}\right)^2. \quad (1)$$

$$\text{This readily gives } x = \left(\frac{h^2 - (2 - \sqrt{2})d^2}{2 + \sqrt{2}}\right)^{\frac{1}{2}}. \quad (2)$$

Having found x , we of course know AO and CO . Let AO be denoted by a , and CO by b ; also let the radius of the inscribed circle be denoted by r .



$$\begin{aligned}\text{Then} \quad \text{AF} &= \sqrt{\text{AO}^2 - \text{OF}^2} = \sqrt{a^2 - r^2}, \\ \text{CE} &= \sqrt{\text{CO}^2 - \text{OE}^2} = \sqrt{b^2 - r^2}.\end{aligned}$$

But $\text{AF} = \text{AG}$, and $\text{CE} = \text{CG}$; therefore,

$$\sqrt{a^2 - r^2} + \sqrt{b^2 - r^2} = h. \quad (3)$$

This readily gives

$$\begin{aligned}r &= \left\{ b^2 - \left(\frac{h^2 + b^2 - a^2}{2h} \right)^2 \right\}^{\frac{1}{2}}, \text{ or} \\ r &= \frac{\sqrt{(a+b+h)(-a+b+h)(a-b+h)(a+b-h)}}{2h}. \quad (4)\end{aligned}$$

This value of r might have been found as follows: In the triangle AOC , all the sides are known; and it is required to find the perpendicular OG , which is r , the radius of the circle. Under P. IV. we have found the perpendiculars when the three sides are known. The perpendicular OG found in this way will be precisely the same as the above value of r .

Now, having found r , we will denote AB by x , and CB by y ; then $\text{AF} = x - r$, and $\text{CE} = y - r$. Hence,

$$\text{AF} + \text{CE} = h = x + y - 2r. \quad (5)$$

Again, $xy = \text{double the area of the triangle } \text{ABC}$; also, $(x + y + h)r = \text{double the area}$. Therefore,

$$xy = (x + y + h)r. \quad (6)$$

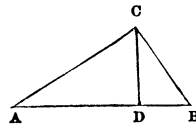
Equations (5) and (6) readily make known x and y , or the sides of the triangle; these values are

$$\left. \begin{aligned}x &= \frac{h + 2r + \sqrt{h^2 - 4hr - 4r^2}}{2}, \\ y &= \frac{h + 2r - \sqrt{h^2 - 4hr - 4r^2}}{2}.\end{aligned} \right\} \quad (7)$$

PROBLEM IX.

In a right-angled triangle, having given the perimeter, and the perpendicular drawn from the right angle upon the hypotenuse; to find the sides of the triangle.

Let $S = \text{sum of sides, or perimeter}$; $P = \text{perpendicular } \text{CD}$; $\text{AC} = x$, and $\text{BC} = y$. Then will $\sqrt{x^2 + y^2} = \text{AB}$. $xy = \text{double the}$



area of triangle ABC; also, $P\sqrt{x^2+y^2}$ = double the area of triangle ABC; therefore,

$$xy = P\sqrt{x^2+y^2}. \quad (1)$$

We also have $x + y + \sqrt{x^2+y^2} = S. \quad (2)$

Equation (2) becomes, by transposing and squaring,

$$(x+y)^2 = (S - \sqrt{x^2+y^2})^2, \text{ or}$$

$$x^2 + 2xy + y^2 = S^2 - 2S\sqrt{x^2+y^2} + x^2 + y^2, \text{ or}$$

$$2xy = S^2 - 2S\sqrt{x^2+y^2}. \quad (3)$$

Taking the double of (1), we have

$$2xy = 2P\sqrt{x^2+y^2}. \quad (4)$$

Equating right-hand members of (3) and (4), we have

$$2P\sqrt{x^2+y^2} = S^2 - 2S\sqrt{x^2+y^2}, \text{ or}$$

$$\sqrt{x^2+y^2} = \frac{S^2}{2(P+S)}. \quad (5)$$

Using this value in (1), we find

$$xy = \frac{PS^2}{2(P+S)}. \quad (6)$$

Squaring (5), we have

$$x^2 + y^2 = \frac{S^4}{4(S+P)^2}. \quad (7)$$

Adding twice (6) to (7), and also subtracting twice (6) from (7), we have

$$x^2 + 2xy + y^2 = \frac{S^2(S^2 + 4PS + 4P^2)}{4(P+S)^2}, \quad (8)$$

$$x^2 - 2xy + y^2 = \frac{S^2(S^2 - 4PS - 4P^2)}{4(P+S)^2}. \quad (9)$$

Extracting the square root of (8) and (9), we have

$$x + y = \frac{S(S+2P)}{2(P+S)}, \quad (10)$$

$$x - y = \frac{S\sqrt{S^2 - 4PS - 4P^2}}{2(P+S)}. \quad (11)$$

Hence,

$$x = \frac{S(S+2P) + S\sqrt{S^2 - 4PS - 4P^2}}{4(P+S)}, \quad (12)$$

$$y = \frac{S(S-2P) - S\sqrt{S^2 - 4PS - 4P^2}}{4(P+S)}. \quad (13)$$

PROBLEM X.

To determine a right-angled triangle; having given the hypotenuse, and side of the inscribed square.

Let h = hypotenuse; s = side of inscribed square; $AB = x$, and $BC = y$.

Then $AE : AB :: EF : BC$;
that is, $x - s : x :: s : y$;

or, $xy = (x + y)s$. (1)

Again, $x^2 + y^2 = h^2$. (2)

Add twice (1) to (2), and we obtain

$$\begin{aligned} x^2 + 2xy + y^2 &= 2s(x + y) + h^2; \text{ or} \\ (x + y)^2 - 2s(x + y) &= h^2, \end{aligned} \quad (3)$$

which is a quadratic in terms of $x + y$: hence we find

$$x + y = s + \sqrt{s^2 + h^2}. \quad (4)$$

Substituting this value of $x + y$ in (1), we shall obtain

$$xy = s^2 + s\sqrt{s^2 + h^2}. \quad (5)$$

From the square of (4) subtracting four times (5), we obtain

$$x^2 - 2xy + y^2 = h^2 - 2s^2 - 2s\sqrt{s^2 + h^2}. \quad (6)$$

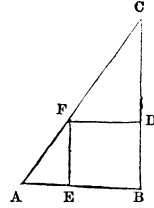
Extracting the square root of (6), we have

$$x - y = \sqrt{h^2 - 2s^2 - 2s\sqrt{s^2 + h^2}}. \quad (7)$$

Taking half the sum of (4) and (7), and also half their difference, we obtain

$$x = \frac{s + \sqrt{s^2 + h^2} + (h^2 - 2s^2 - 2s\sqrt{s^2 + h^2})^{\frac{1}{2}}}{2}, \quad (8)$$

$$y = \frac{s + \sqrt{s^2 + h^2} - (h^2 - 2s^2 - 2s\sqrt{s^2 + h^2})^{\frac{1}{2}}}{2}. \quad (9)$$

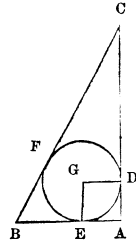


PROBLEM XI.

To determine a right-angled triangle; having given the hypotenuse, and the radius of the inscribed circle.

Let h = hypotenuse; r = radius of the inscribed circle;
circle; $AB = x$, and $AC = y$.

Then $BF = BE = x - r$,
and $CF = CD = y - r$;
therefore $BC = BF + CF = x + y - 2r = h$,
or, $x + y = 2r + h$. (1)



Again, xy = double the area;

$$(x + y + h)r = (x + y)r + hr = \text{double the area:}$$

$$\text{therefore } xy = (x + y)r + hr. \quad (2)$$

In (2), for $x + y$ substitute its value given by (1), and it will become

$$xy = 2r(r + h). \quad (3)$$

Equations (1) and (3) readily give

$$x = \frac{2r + h + \sqrt{h^2 - 4hr - 4r^2}}{2}, \quad (4)$$

$$y = \frac{2r + h - \sqrt{h^2 - 4hr - 4r^2}}{2}. \quad (5)$$

PROBLEM XII.

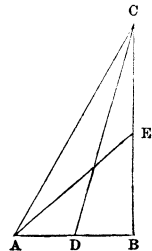
To determine a right-angled triangle; having given the lengths of the lines drawn from the acute angles to the middle points of the opposite sides.

Let $AE = a$, and $CD = b$; also $AB = x$, and $BC = y$. Then will

$$\begin{aligned} AB^2 + BE^2 &= a^2, \text{ or} \\ x^2 + \frac{1}{4}y^2 &= a^2. \end{aligned} \quad (1)$$

In a similar way we find

$$y^2 + \frac{1}{4}x^2 = b^2. \quad (2)$$



Equations (1) and (2), when cleared of fractions, become

$$4x^2 + y^2 = 4a^2, \quad (3)$$

$$x^2 + 4y^2 = 4b^2. \quad (4)$$

These equations give

$$x = 2\sqrt{\frac{4a^2 - b^2}{15}}, \quad (5)$$

$$y = 2\sqrt{\frac{4b^2 - a^2}{15}}. \quad (6)$$

PROBLEM XIII.

To determine a triangle; having given the base, the perpendicular, and the difference of the two other sides.

Let p = the perpendicular; d = half the difference of sides; $x + d = AC$; $x - d = BC$.
 b = half the base; $b + y = AD$; $b - y = DB$.

Then, from the right-angled triangles ADC, BDC, we have

$$(b + y)^2 + p^2 = (x + d)^2, \quad (1)$$

$$(b - y)^2 + p^2 = (x - d)^2. \quad (2)$$

Expanding (1) and (2), and then taking half their sum and one fourth of their difference, we obtain

$$b^2 + y^2 + p^2 = x^2 + d^2, \quad (3)$$

$$by = dx. \quad (4)$$

These equations readily give

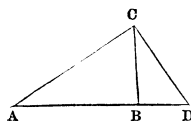
$$x = b \left(\frac{b^2 - d^2 + p^2}{b^2 - d^2} \right)^{\frac{1}{2}}, \quad (5)$$

$$y = d \left(\frac{b^2 - d^2 + p^2}{b^2 - d^2} \right)^{\frac{1}{2}}. \quad (6)$$

Hence,

$$AC = b \left(\frac{b^2 - d^2 + p^2}{b^2 - d^2} \right)^{\frac{1}{2}} + d, \quad (7)$$

$$BC = b \left(\frac{b^2 - d^2 + p^2}{b^2 - d^2} \right)^{\frac{1}{2}} - d. \quad (8)$$



PROBLEM XIV.

To determine the radii of three equal circles, described in a given circle, to touch each other, and also to touch the circumference of the given circle.

Let D be the centre of the given circle, whose radius we will denote by R ; also, let A, B, C be the centres of the three equal circles, whose common radius we will denote by r . Then joining A, B, and C, we have the triangle ABC equilateral, each of whose sides is $2r$. Drawing CK perpendicular to AB, the right-angled triangle AKC gives

$$CK^2 = AC^2 - AK^2,$$

$$\text{or, in symbols, } CK^2 = 4r^2 - r^2 = 3r^2;$$

$$\text{consequently, } CK = r\sqrt{3}. \quad (1)$$

Now (B. I., T. XXXVIII.),

$$CD = \frac{2}{3}CK = \frac{2}{3}r\sqrt{3}. \quad (2)$$

But

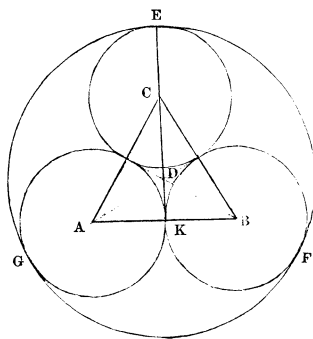
$$ED = EC + CD;$$

that is,

$$R = r + \frac{2}{3}r\sqrt{3}; \quad (3)$$

from which we readily find

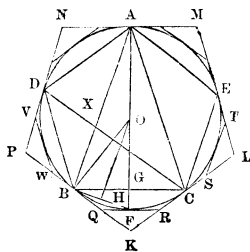
$$r = \frac{3R}{3 + 2\sqrt{3}} = \frac{R}{1 + 2\sqrt{\frac{1}{3}}}. \quad (4)$$



PROBLEM XV.

Given the radius, r , of a circle, to find the sides of the inscribed and circumscribed pentagons and decagons.

1. *The inscribed pentagon.* Let ADBCE be the pentagon inscribed in the circle, and let O be the centre of the circumscribing circle. Join AB, AC, and draw AF perpendicular to BC; then, by known properties, BC is bisected in G, and the line AF passes through the centre O of the circle.



Since the angle ADX is measured by half the arc AEC (B. II., T. X.), and the angle AXD is measured by half the sum of the arcs AD and BC (B. II., T. XI.), it follows that these angles are equal, and the triangle DAX is isosceles.

Again, the angles DAX and BAC are equal, being measured by halves of the equal arcs DB, BC; hence the triangles ABC and DAX are similar. But the triangle DAX is obviously equal to BCX; therefore,

$$AB : BC :: BC : BX;$$

but $BX = AB - AX = AB - BC;$

therefore, $AB : BC :: BC : AB - BC.$ (1)

Put $BC = 2x$, or $BG = x$; $BA = y$, and $BF = z$; and the above proportion will become

$$y : 2x :: 2x : y - 2x;$$

which gives

$$y = (1 + \sqrt{5})x. \quad (2)$$

Now, we have $OG = \sqrt{r^2 - x^2}$, $AG = \sqrt{y^2 - x^2}$; and hence $AG = AO + OG$ gives

$$\sqrt{y^2 - x^2} = r + \sqrt{r^2 - x^2},$$

or, squaring, $y^2 - 2r^2 = 2r\sqrt{r^2 - x^2}.$

Again, squaring, we have

$$y^4 - 4r^2y^2 + 4r^2x^2 = 0. \quad (3)$$

Substituting the value of y as given by (2), we find the side of an inscribed pentagon (see B. IV., T. XI., S. II.),

$$2x = \frac{1}{2}r\sqrt{10 - 2\sqrt{5}}. \quad (4)$$

We have $y = x(1 + \sqrt{5}) = \frac{1}{2}r\sqrt{10 + 2\sqrt{5}},$ (5)

$$OG = \sqrt{r^2 - x^2} = \sqrt{r^2 - \frac{1}{8}(5 - \sqrt{5})r^2} = \frac{1}{4}r(1 + \sqrt{5}). \quad (6)$$

2. *The inscribed decagon.* Join BF; then, since AF bisects the line BC at right angles, it bisects the arc BFC in F; and hence BF is the side of the inscribed decagon. But ABF being a right angle, since it is in a semicircle, we have

$$BF^2 = FA^2 - AB^2; \text{ or, in symbols,}$$

$$z^2 = 4r^2 - y^2 = 4r^2 - \frac{1}{4}(10 + 2\sqrt{5})r^2 = \frac{1}{2}(3 - \sqrt{5})r^2;$$

or, extracting the square root, we have, for the side of the inscribed decagon (see B. IV., T. XI., S. I.),

$$z = \frac{1}{2}r(\sqrt{5} - 1). \quad (7)$$

3. *The circumscribing pentagon.* The inscribed and circumscribed pentagons being regular, are similar figures, and their sides are as the perpendiculars from the centre upon the sides. That is, if PK be a side of the circumscribing pentagon, we shall have

OG : OB :: BC : PK; or, in symbols,

$$PK = \frac{OB \times BC}{OG} = \frac{r \cdot \frac{1}{2}r \sqrt{10 - 2\sqrt{5}}}{\frac{1}{4}r(1 + \sqrt{5})} = 2r\sqrt{5 - 2\sqrt{5}}. \quad (8)$$

4. *The circumscribing decagon.* Let QR be one of the sides; and draw OH perpendicular to BF, which it bisects in H. Also by similar triangles ABF, OHF, we have

$$OH = \frac{1}{2}AB = \frac{1}{2}y = \frac{1}{4}r \sqrt{10 + 2\sqrt{5}}.$$

Also, as in the last case,

OH : OF :: BF : QR; which gives

$$QR = \frac{OF \times BF}{OH} = \frac{r \cdot \frac{1}{2}r(\sqrt{5} - 1)}{\frac{1}{4}r \sqrt{10 + 2\sqrt{5}}} = 2r\sqrt{\frac{5 - 2\sqrt{5}}{5}}.$$

PROBLEM XVI.

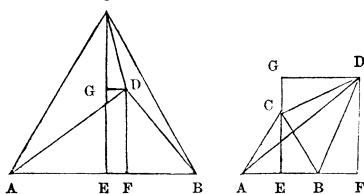
Given the lengths of three lines drawn from a point to the three angles of an equilateral triangle, to find its side.

Let ABC be the triangle, and D the point.

Put $AD = a,$
 $BD = b,$
 $CD = c;$
 $AB = 2x,$
 $EF = y,$
 $FD = z.$

Then will $AF = x + y,$

and $BF = x - y$ or $y - x.$



$$CE = \sqrt{AC^2 - AE^2} = \sqrt{4x^2 - x^2} = x\sqrt{3};$$

$$CG = x\sqrt{3} - z, \text{ or } z - x\sqrt{3}.$$

Hence we have the following relations true, whether the point D is *within* the equilateral triangle, or *without* it.

$$(x + y)^2 + z^2 = a^2, \quad (1)$$

$$(x - y)^2 + z^2 = b^2, \quad (2)$$

$$(x\sqrt{3} - z)^2 + y^2 = c^2. \quad (3)$$

Subtracting (2) from (1), we find

$$4xy = a^2 - b^2, \text{ or } y = \frac{a^2 - b^2}{4x}; \quad (4)$$

consequently,
$$y^2 = \frac{(a^2 - b^2)^2}{16x^2}. \quad (5)$$

Equation (1) gives immediately

$$z = \sqrt{a^2 - (x + y)^2};$$

in which, substituting for y its value given by (4), we have

$$z = \sqrt{a^2 - \left(x + \frac{a^2 - b^2}{4x}\right)^2}. \quad (6)$$

This value of z , and the value of y^2 given by (5), being substituted in (3), will cause it to become

$$\left\{ x\sqrt{3} - \sqrt{a^2 - \left(x + \frac{a^2 - b^2}{4x}\right)^2} \right\}^2 + \frac{(a^2 - b^2)^2}{16x^2} = c^2. \quad (7)$$

This equation contains only the unknown x , which value may therefore be found: the reduction leads to

$$16x^4 - 4(a^2 + b^2 + c^2)x^2 = -a^4 - b^4 - c^4 + a^2b^2 + b^2c^2 + c^2a^2. \quad (8)$$

This equation, when solved by the rule for quadratics, gives

$$2x = \left\{ \frac{a^2 + b^2 + c^2 \pm \sqrt{6(a^2b^2 + b^2c^2 + c^2a^2) - 3(a^4 + b^4 + c^4)}}{2} \right\}^{\frac{1}{2}}. \quad (9)$$

We must use the + sign when the point is within the triangle, as in the first figure; and the - sign when the point is without, as in the second figure.

If we suppose a triangle to be formed with the three lines a , b , c , and denote its area by Δ , expression (9) will become

$$2x = \left(\frac{a^2 + b^2 + c^2 \pm 4\Delta\sqrt{3}}{2} \right)^{\frac{1}{2}}. \quad (10)$$

PROBLEM XVII.

Suppose AB to be a tree, standing on the horizontal plane AC; it is required to find at what point it must break, so that, by falling, the top may strike the ground at C.

If we join BC, and bisect it with the perpendicular DF, we shall have F for the point sought.

Let the height of the tree be denoted by h , and the distance AC by b ; then BC will equal

$$\sqrt{b^2 + h^2};$$

and, consequently,

$$CD = \frac{1}{2} BC = \frac{1}{2} \sqrt{b^2 + h^2}.$$

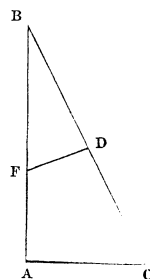
Again, the two triangles BDF and BAC are similar, having the angle B common, and the angle BDF equal to the angle BAC, each being a right angle; therefore,

$$BA : BC :: BD : BF,$$

$$h : \sqrt{b^2 + h^2} :: \frac{1}{2} \sqrt{b^2 + h^2} : BF.$$

Consequently,
$$BF = \frac{b^2 + h^2}{2h},$$

and
$$AF = AB - BF = h - \frac{b^2 + h^2}{2h} = \frac{h^2 - b^2}{2h}.$$



PROBLEM XVIII.

Suppose AC and BD to represent two trees standing on the horizontal plane AB; it is required to find a point in this plane, situated on the line AB, equally distant from the tops C and D.

Join CD, and bisect it with the perpendicular FG; then G is the point sought.

This point may be found arithmetically as follows:

Draw DK parallel to AB, and FL perpendicular to AB. Let the height of the tree AC be denoted by h_1 , the height of the tree BD by h_2 , and the distance AB by d ; then will

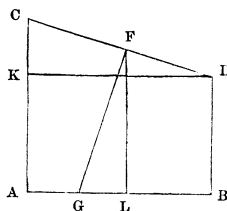
$$KC = AC - BD = h_1 - h_2, \quad FL = \frac{1}{2}(h_1 + h_2).$$

Since the sides of the two triangles DKC, FLG are respectively perpendicular to each other, they are similar, and we have

$$DK : KC :: FL : LG;$$

or, in symbols, $d : h_1 - h_2 :: \frac{1}{2}(h_1 + h_2) : LG.$

Consequently,
$$LG = \frac{h_1^2 - h_2^2}{2d};$$



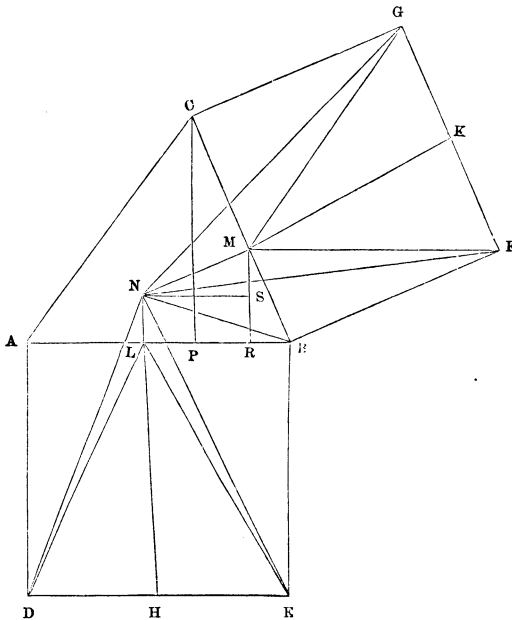
and $AG = AL - LG = \frac{1}{2}d - \frac{h_1^2 - h_2^2}{2d} = \frac{d^2 - h_1^2 + h_2^2}{2d},$

$BG = BL + LG = \frac{1}{2}d + \frac{h_1^2 - h_2^2}{2d} = \frac{d^2 + h_1^2 - h_2^2}{2d}.$

PROBLEM XIX.

Suppose three trees to stand vertically upon a horizontal plane at the points A, B, and C. It is required to find a point in this plane equally distant from their tops.

Pass a plane through the two trees at A and B, and suppose this plane to revolve about the line AB until it coincides with the horizontal plane, the trees being then represented by AD and BE. Join DE and bisect it with the perpendicular HL;



then will the point L be equally distant from the points D and E. Also, pass a plane through the two trees at B and C, and suppose this plane to be revolved about the line BC until it coincides with the horizontal plane, the trees then taking the position of BF and CG. Join FG and bisect it with the per-

pendicular KM; then will the point M be equally distant from the points F and G.

Draw LN and MN respectively perpendicular to AB and BC, meeting at the point N. Then will N be the point sought.

For, suppose the planes ADEB and BFGC to be brought back to the first position, so as to be perpendicular to the horizontal plane ABC. Then join ND, NE, NF, and NG; also, join LD, LE, MF, and MG. Since BE is equal to BF, each representing the height of the tree standing at B, it follows that NE is equal to NF. NL will be perpendicular to the plane ADEB, consequently the triangles DLN, ELN are right-angled, and DN and LN are respectively equal to EL and LN; hence their hypotenuses DN and EN are equal, which proves N to be equally distant from the tops of the trees standing at A and B.

In a similar manner it may be shown that the right-angled triangles FMN, GMN are equal, and consequently the side FN is equal to GN. But FN has already been shown to be equal to EN, consequently DN, EN, FN, and GN are each of the same length; that is, N is a point in the horizontal plane equally distant from the tops of the three trees standing at the points A, B, and C.

As an illustration, let us suppose $AD = 114$ feet, $BE = 110$ feet $= BF$, $CG = 98$ feet. Also, suppose $AB = 112$, $BC = 104$, $AC = 120$. Then we find $BL = 60$, $BM = 40$ (see P. XVIII.).

The perpendicular CP we find (P. IV.) to be 96, and $BP = 40$. If we draw MR perpendicular to AB, and NS perpendicular to MR, we shall have the triangles CPB, MRB, and NSM right-angled and similar. Hence, by the similar triangles CPB, MRB, we have

$$BC : BM :: BP : BR, \text{ or}$$

$$104 : 40 :: 40 : BR = 15\frac{5}{13}.$$

And $LR = BL - BR = 60 - 15\frac{5}{13} = 44\frac{8}{13}.$

Again, by the similar triangles CPB, NSM, we have

$$CP : CB :: NS (= LR) : NM, \text{ or}$$

$$96 : 104 :: 44\frac{8}{13} : NM = 48\frac{1}{3}.$$

Since the triangle NMB is right-angled, we have

$$BN^2 = NM^2 + BM^2 = 2336\frac{1}{9} + 1600 = 3936\frac{1}{9}.$$

Again, since the triangle NBE, or its equal NBF, is right-angled at B, we have

$$NE^2 = NF^2 = BN^2 + BE^2 = 3936\frac{1}{9} + 12100 = 16036\frac{1}{9}.$$

And $NE = NF = NG = ND = \sqrt{16036\frac{1}{9}} = 126.63$, nearly, for the number of feet from the point N to the top of each tree.

END OF PART I.

PART SECOND.

PLANE AND SPHERICAL
TRIGONOMETRY,
ETC.

TRIGONOMETRY.

CHAPTER I.

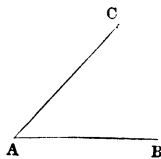
FIRST PRINCIPLES OF PLANE TRIGONOMETRY.

§ 1. TRIGONOMETRY is that science which investigates the relations of the sides and angles of triangles.

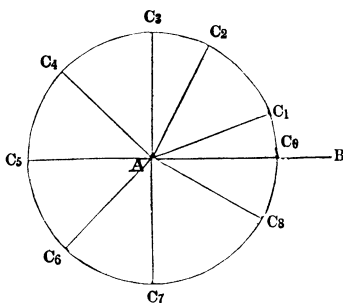
When confined to plane triangles, it is called *Plane Trigonometry*.

Definition of an Angle.

§ 2. When two straight lines, AB, AC, meet each other, the opening between these lines is called an angle. This angle or opening, which is read BAC or CAB, or simply the angle at A, does not depend upon the lengths of these lines, but only upon the difference of their directions.



§ 3. If any straight line, as AB, revolve about A as a centre, in such a manner that the part AC_0 shall take the successive positions AC_1, AC_2, AC_3 , &c., then will the arc described by the point C_0 increase at the same rate as the angle increases; so that the arc C_0C_1 is to the arc C_0C_2 , as the angle C_0AC_1 is to the angle C_0AC_2 ; and the same for other arcs and their corresponding angles. Hence, the arcs thus described may be taken as measures of their corresponding angles.



§ 4. When the line AC_0 has reached the position AC_3 , so that the arc C_0C_3 is a *quadrant*, or one-fourth of the entire cir-

cumference, the angle C_0AC_3 measured by this quadrant, is called a *right angle*. The right angle is usually taken as the unit angle; and all angles less than a right angle are called *acute angles*, and are each measured by an arc less than one-fourth of an entire circumference.

When the line AC_0 has reached the position AC_5 , the point C_0 has passed over an arc equal to one-half of the entire circumference, or over an arc which is the measure of two right angles. Hence, when two lines meet from opposite directions, they are said to form an angle equal to two right angles.

An angle such as C_0AC_4 , which is greater than a right angle and less than two right angles, is called an *obtuse angle*.

§ 5. All the foregoing relations hold good, whatever be the length of the radius AC_0 . For simplicity, we shall hereafter use such arcs, for the measure of their corresponding angles, as are described with a *unit* radius.

The Greek letter π is usually employed to denote the semi-circumference of a circle whose radius is a unit, or, which is the same thing, it represents the ratio of any circumference to its diameter. Hence $\frac{\pi}{2}$ is the length of an arc called a quadrant, which is the measure of a right angle. The numerical value of π is as follows :

$\pi = 3.1415926535897932384$, &c. = semi-circumference, when the radius = 1. (See Geom. B. IV., Lemma.)

§ 6. If the whole circumference be divided into 360 equal parts, one of those parts is called a *degree*. Degrees are also divided into 60 equal parts, called *minutes*; minutes are similarly divided into 60 equal parts, called *seconds*, and so on for other subdivisions. Thus $43^\circ 14' 12''$ denotes an angle whose measuring arc consists of 43 degrees, 14 minutes, and 12 seconds. In the same way 90° denotes an angle whose measuring arc consists of $\frac{1}{4}$ of 360 degrees, and whose length is therefore $\frac{\pi}{2} = 1.5707963267948966192$, &c. From the above value of π , which is the arc of 180° , it is easy to find the length of an arc corresponding to any angle expressed in degrees, minutes, and seconds. Thus, we find,

$$\begin{aligned} \frac{1}{180} \text{ of } \pi = 1^\circ &= 0.01745329251994329577, \text{ \&c.} \\ \frac{1}{60} \text{ of } \frac{1}{180} \text{ of } \pi = 1' &= 0.00029088820866572160, \text{ \&c.} \\ \frac{1}{60} \text{ of } \frac{1}{60} \text{ of } \frac{1}{180} \text{ of } \pi = 1'' &= 0.00000484813681109536, \text{ \&c.} \end{aligned}$$

Modern French writers, instead of using the *sexagesimal* division, use the *centesimal*; and it is to be regretted that the latter is not universally used, on account of the great ease with which all calculations are made in that division.

If E denote the number of English degrees in a given angle, and F denote the number of French grades in the same angle, we shall obviously have

$$\frac{E}{90} = \frac{F}{100}, \text{ or } \frac{E}{9} = \frac{F}{10}. \text{ Consequently,}$$

$$E = \frac{9}{10} \text{ of } F = F - \frac{1}{10} \text{ of } F; \text{ and } F = \frac{10}{9} \text{ of } E = E + \frac{1}{9} \text{ of } E.$$

§ 7. When the sum of two angles is equal to a right angle, those angles are mutually *complements* of each other. In such

case, the sum of their measuring arcs is $\frac{\pi}{2}$, and the sum of the

degrees which these arcs contain is 90° . When the sum of two angles is equal to two right angles, these angles are mutually *supplement* of each other: the sum of the arcs which measure these angles is equal to π , and

the sum of the degrees which they contain is 180° . Thus the

angles C_0AC_1 , C_1AC_3 , are mutually complements of each other.

Also if we denote any arc by a , its complement will be $\frac{\pi}{2} - a$,

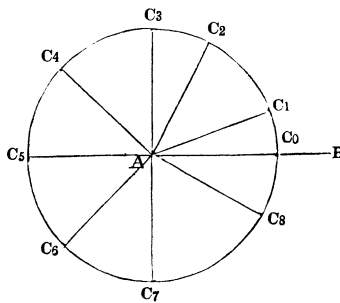
and its supplement will be $\pi - a$.

If an angle is expressed in degrees, as A° , where A° denotes

the number of degrees in the arc which measures the given angle, then will $90^\circ - A^\circ$ denote the number of degrees in the arc which measures its complement. For brevity, we say that

a and $\frac{\pi}{2} - a$ are complementary arcs, and that A° and $90^\circ - A^\circ$

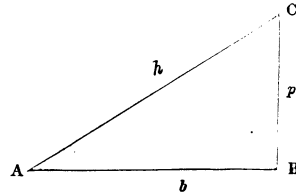
are complementary degrees. For a like reason, a and $\pi - a$ are supplementary arcs, and A° and $180^\circ - A^\circ$ are supplementary degrees.



Definition of Sines, Tangents, Secants, &c.

§ 8. In a right triangle the *sine* of either of the acute angles is the quotient obtained by dividing the side opposite the given angle by the hypotenuse.

Thus, in the right triangle ABC, if we represent the base, perpendicular, and hypotenuse, by b , p , and h respectively, we shall have



$$\text{sine } A = \frac{p}{h}; \text{ sine } C = \frac{b}{h}.$$

The *tangent* of either of the acute angles is the quotient obtained by dividing the side opposite the angle by the adjacent side. Thus,

$$\text{tangent } A = \frac{p}{b}; \text{ tangent } C = \frac{b}{p}.$$

The *secant* of either of the acute angles is the quotient obtained by dividing the hypotenuse by the side adjacent the given angle. Thus,

$$\text{secant } A = \frac{h}{b}; \text{ secant } C = \frac{h}{p}.$$

The *cosine*, *cotangent*, and *cosecant* of any angle are respectively the sine, tangent, and secant of its complement. The two acute angles of a right triangle being complements of each other, it follows that we shall have these relations :

$$\left. \begin{array}{lll} \sin. & A = \cos. & C = \frac{p}{h}. \quad (1) \\ \cos. & A = \sin. & C = \frac{b}{h}. \quad (2) \\ \text{tang.} & A = \cotan. & C = \frac{p}{b}. \quad (3) \\ \cotan. & A = \text{tang.} & C = \frac{b}{p}. \quad (4) \\ \text{sec.} & A = \text{cosec.} & C = \frac{h}{b}. \quad (5) \\ \text{cosec.} & A = \text{sec.} & C = \frac{h}{p}. \quad (6) \end{array} \right\} \dots (A).$$

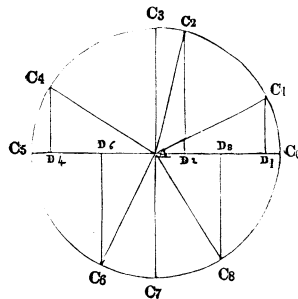
From the above relations, we immediately find

$$\left. \begin{aligned} \frac{\sin. A}{\cos. A} &= \frac{p}{b} = \text{tang. } A. & (1) \\ \frac{\cos. A}{\sin. A} &= \frac{b}{p} = \text{cot. } A. & (2) \\ \frac{1}{\cos. A} &= \frac{h}{b} = \text{sec. } A. & (3) \\ \frac{1}{\sin. A} &= \frac{h}{p} = \text{cosec. } A. & (4) \\ \sin.^2 A + \cos.^2 A &= \frac{p^2 + b^2}{h^2} = 1 & (5) \end{aligned} \right\} \dots (B).$$

The exponent 2, equation (5) of (B), which is used to denote the square of the *sine* and *cosine* of the angle A , is placed immediately after *sin.* and *cos.*, so that $\sin.^2 A$ is the same as $(\sin. A)^2$, $\cos.^2 A$ is the same as $(\cos. A)^2$.

From the above relations, we see that the *sine* and *cosecant* are reciprocals of each other; *cosine* and *secant* are also reciprocals; so also are the *tangent* and *cotangent*.

§ 9. If from one extremity of the arc which measures an angle, a perpendicular be drawn to the diameter which passes through the other extremity, such perpendicular will be the sine of that angle. Thus, C_1D_1 is the sine of the angle C_0AC_1 . For we have (§ 8) defined the sine of this angle to be the quotient obtained by dividing C_1D_1 by AC_1 , which quotient becomes C_1D_1 , since the divisor is the radius of the circle, and therefore (§ 5) equal to unity. In the same way it may be shown that C_2D_2 is the sine of the angle C_0AC_2 , that C_4D_4 is the sine of the angle C_0AC_4 . As the arc is the measure of angles, we say that C_1D_1 , C_2D_2 , C_4D_4 , are respectively the sines of the arcs C_0C_1 , C_0C_2 , C_0C_4 .

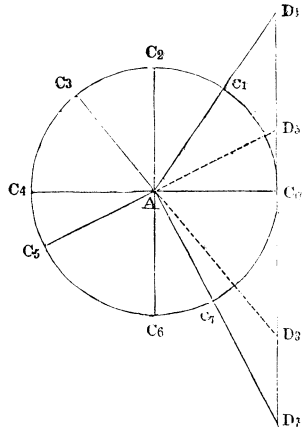


Passing from the angle to the arc, we also say, when the arc exceeds the semi-circumference, that C_6D_6 , C_8D_8 , are respectively the sines of the arcs $C_0C_5C_6$, $C_0C_5C_7C_8$.

In all these cases we suppose the arc to be estimated from the point C_0 , at the extreme right of the circle, upwards and around towards the left. When angles are estimated from C_0 downwards around towards the left, they are considered as negative arcs. Thus, C_0AC_8 is considered as a negative angle, measured by the negative arc C_0C_8 . By reference to the diagram, it will be readily seen that so long as a positive arc does not exceed π or 180° , the sines are all situated above the horizontal diameter, and are considered as positive; for arcs greater than 180° and less than 360° , their sines are situated below the horizontal diameter, and are considered as negative.

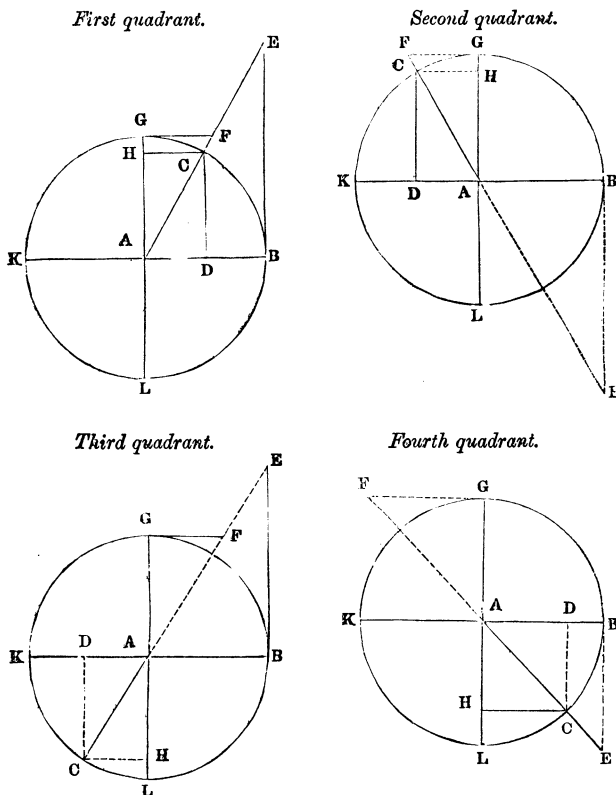
§ 10. If from one extremity of an arc a tangent line be drawn, and limited by its intersection with the radius drawn through the other extremity of the arc, then will this portion of the tangent line thus included between the radius and the radius produced, be the trigonometrical tangent of the arc. And the radius thus produced, or simply the produced part, will be the secant of the arc. Thus, C_0D_1 is the tangent of the arc C_0C_1 , and C_0D_3 is the tangent of the arc C_0C_3 , in the same way C_0D_5 , C_0D_7 are respectively the tangents of the arcs $C_0C_2C_5$, $C_0C_2C_4C_7$. For, by the former definition (§ 8), the tangent of the angle C_0AD_1 , which is measured by the arc C_0C_1 , is the quotient obtained by dividing C_0D_1 by AC_0 , which quotient becomes C_0D_1 , since AC_0 , the radius, is a unit. Also the secant of this angle is the quotient obtained by dividing AD_1 by AC_0 , which quotient becomes AD_1 . When the arc terminates in the first and third quadrants, the tangents are counted on the line D_7D_1 from C_0 upwards, and are considered as positive. But in the second and fourth quadrants, the tangents are counted from C_0 downwards, and are considered as negative.

The secants of the arcs C_0C_1 , C_0C_3 , $C_0C_2C_5$, $C_0C_2C_4C_7$, are respectively AD , AD_3 , AD_5 , AD_7 . By inspecting the diagram it will be seen that the secants represented by the dotted lines



consist of the produced part of the radius, while those represented by the full lines consist of the radius together with the produced part; the latter are considered as positive and the former as negative. Hence in the first and fourth quadrants the secants are positive, and in the second and third quadrants they are negative.

§ 11. These different lines may be clearly exhibited in the four quadrants as follows :



In the above diagrams the dotted lines are considered as negative and the full lines as positive.

The letters in the four diagrams are so arranged that in each case BC counted in the positive direction (§ 9) is the arc, CD the sine, CH the cosine, BE the tangent, GF the cotangent, AE the secant, and AF the cosecant.

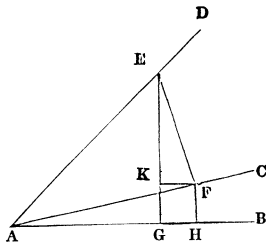
In general, if n denote any integral number, including also the value of $n=0$, we shall have,

$$\begin{aligned}
 \sin. \quad 0^\circ &= \sin. 180^\circ = \sin. 2n \times 90^\circ = 0 \\
 \sin. \quad 90^\circ &= \sin. (4n+1) \times 90^\circ = 1 \\
 \sin. \quad 270^\circ &= \sin. (4n+3) \times 90^\circ = -1 \\
 \cos. \quad 0^\circ &= \cos. 4n \times 90^\circ = 1 \\
 \cos. \quad 90^\circ &= \cos. 270^\circ = \cos. (2n+1) \times 90^\circ = 0 \\
 \cos. \quad 180^\circ &= \cos. (4n+2) \times 90^\circ = -1 \\
 \text{tang.} \quad 0^\circ &= \text{tang. } 180^\circ = \text{tang. } 2n \times 90^\circ = 0 \\
 \text{tang.} \quad 90^\circ &= \text{tang. } 270^\circ = \text{tang. } (2n+1) \times 90^\circ = \infty \\
 \text{cotan.} \quad 0^\circ &= \text{cotan. } 180^\circ = \text{cotan. } 2n \times 90^\circ = \infty \\
 \text{cotan.} \quad 90^\circ &= \text{cotan. } 270^\circ = \text{cotan. } (2n+1) \times 90^\circ = 0 \\
 \sec. \quad 0^\circ &= \sec. 4n \times 90^\circ = 1 \\
 \sec. \quad 90^\circ &= \sec. 270^\circ = \sec. (2n+1) \times 90^\circ = \infty \\
 \sec. \quad 180^\circ &= \sec. (4n+2) \times 90^\circ = -1 \\
 \text{cosec.} \quad 0^\circ &= \text{cosec. } 180^\circ = \text{cosec. } 2n \times 90^\circ = \infty \\
 \text{cosec.} \quad 90^\circ &= \text{cosec. } (4n+1) \times 90^\circ = 1 \\
 \text{cosec.} \quad 270^\circ &= \text{cosec. } (4n+3) \times 90^\circ = -1
 \end{aligned}$$

§ 15. *To find the sine and cosine of the sum and difference of two arcs.*

Let the angle $BAC=a$; that is, let a denote the length of the arc which measures this angle; also let the angle CAD be denoted by b ; then will BAD be denoted by $a+b$.

Draw EF perpendicular to AC , and from the points E and F draw EG and FH perpendicular to AB ; also draw FK perpendicular to EG . Since the sides of the triangle EFK are respectively perpendicular to the sides of the triangle AFH , these triangles are similar. (Geom. B. III., T. VII.) Hence, the angle FEK is equal to FAH , equal to a .



$$\begin{aligned}
 \sin. (a+b) &= \frac{EG}{AE} = \frac{HF + KE}{AE} = \frac{HF}{AE} + \frac{KE}{AE} = \\
 &= \frac{HF}{AF} \times \frac{AF}{AE} + \frac{KE}{EF} \times \frac{EF}{AE}.
 \end{aligned}$$

By our definitions (§ 8), $\frac{HF}{AF} = \sin. a$; $\frac{AF}{AE} = \cos. b$; $\frac{KE}{EF} = \cos. a$; $\frac{EF}{AE} = \sin. b$. Hence, we have

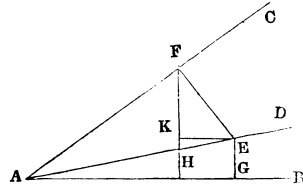
$$\sin. (a + b) = \sin. a \cos. b + \cos. a \sin. b.$$

Again, we have

$$\begin{aligned} \cos. (a + b) &= \frac{AG}{AE} = \frac{AH - KF}{AE} = \frac{AH}{AE} - \frac{KF}{AE} = \\ &= \frac{AH}{AF} \times \frac{AF}{AE} - \frac{KF}{EF} \times \frac{EF}{AE} = \cos. a \cos. b - \sin. a \sin. b. \end{aligned}$$

For the sine and cosine of $a - b$, let the angle $BAC = a$, and the angle $CAD = b$; then will $BAD = a - b$.

Then, as in the former diagram, we have the triangle EFK similar to the triangle AFH , and the angle KFE equal to the angle FAH , equal to a .



$$\begin{aligned} \sin. (a - b) &= \frac{EG}{AE} = \frac{HF - KF}{AE} = \frac{HF}{AF} \times \frac{AF}{AE} - \frac{KF}{FE} \times \frac{FE}{AE} = \\ &= \sin. a \cos. b - \cos. a \sin. b. \end{aligned}$$

$$\begin{aligned} \cos. (a - b) &= \frac{AG}{AE} = \frac{AH + KE}{AE} = \frac{AH}{AF} \times \frac{AF}{AE} + \frac{KE}{FE} \times \frac{FE}{AE} = \\ &= \cos. a \cos. b + \sin. a \sin. b. \end{aligned}$$

Collecting these results, we have

$$\sin. (a + b) = \sin. a \cos. b + \cos. a \sin. b. \quad (1.)$$

$$\sin. (a - b) = \sin. a \cos. b - \cos. a \sin. b. \quad (2.)$$

$$\cos. (a + b) = \cos. a \cos. b - \sin. a \sin. b. \quad (3.)$$

$$\cos. (a - b) = \cos. a \cos. b + \sin. a \sin. b. \quad (4.)$$

If we make $b = a$, equations (1) and (3) will give

$$\sin. 2a = 2 \sin. a \cos. a, \quad (5.)$$

$$\cos. 2a = \cos.^2 a - \sin.^2 a. \quad (6.)$$

From (5) of (B), § 8, we have

$$1 = \sin.^2 a + \cos.^2 a. \quad (7.)$$

The sum and difference of (6) and (7) give

$$1 + \cos. 2a = 2 \cos.^2 a, \quad . \quad . \quad . \quad (8.)$$

$$1 - \cos. 2a = 2 \sin.^2 a. \quad . \quad . \quad . \quad (9.)$$

Hence, $\sin. a = \frac{1}{2} \sqrt{2 - 2 \cos. 2a}, \quad . \quad . \quad (10.)$

$$\cos. a = \frac{1}{2} \sqrt{2 + 2 \cos. 2a}. \quad . \quad . \quad (11.)$$

Equations (5) and (6) make known the sine and cosine of a double arc in terms of the sine and cosine of the arc itself. Equations (10) and (11) give, conversely, the sine and cosine of half an arc in terms of the cosine of the arc itself.

Dividing (1) by (3), we have

$$\frac{\sin. (a+b)}{\cos. (a+b)} = \frac{\sin. a \cos. b + \cos. a \sin. b}{\cos. a \cos. b - \sin. a \sin. b}.$$

Dividing numerator and denominator of the right-hand member each by $\cos. a \cos. b$, we have

$$\frac{\sin. (a+b)}{\cos. (a+b)} = \frac{\frac{\sin. a}{\cos. a} + \frac{\sin. b}{\cos. b}}{1 - \frac{\sin. a}{\cos. a} \times \frac{\sin. b}{\cos. b}},$$

which becomes (§ 8),

$$\text{tang. } (a+b) = \frac{\text{tang. } a + \text{tang. } b}{1 - \text{tang. } a \text{ tang. } b}. \quad . \quad . \quad (12.)$$

§ 16. Equations (1), (2), (3), and (4) (§ 15), give as follows :

$$\sin. (a+b) + \sin. (a-b) = 2 \sin. a \cos. b. \quad (1.)$$

$$\sin. (a+b) - \sin. (a-b) = 2 \cos. a \sin. b. \quad (2.)$$

$$\cos. (a-b) + \cos. (a+b) = 2 \cos. a \cos. b. \quad (3.)$$

$$\cos. (a-b) - \cos. (a+b) = 2 \sin. a \sin. b. \quad (4.)$$

If in these results we substitute p for $a+b$, and q for $a-b$, which gives $a = \frac{1}{2} (p+q)$ and $b = \frac{1}{2} (p-q)$, they will become

$$\sin. p + \sin. q = 2 \sin. \frac{1}{2} (p+q) \cos. \frac{1}{2} (p-q). \quad (5.)$$

$$\sin. p - \sin. q = 2 \cos. \frac{1}{2} (p+q) \sin. \frac{1}{2} (p-q). \quad (6.)$$

$$\cos. q + \cos. p = 2 \cos. \frac{1}{2} (p+q) \cos. \frac{1}{2} (p-q). \quad (7.)$$

$$\cos. q - \cos. p = 2 \sin. \frac{1}{2} (p+q) \sin. \frac{1}{2} (p-q). \quad (8.)$$

Dividing (5) by (6), we have

$$\frac{\sin. p + \sin. q}{\sin. p - \sin. q} = \frac{\sin. \frac{1}{2} (p + q)}{\cos. \frac{1}{2} (p + q)} \times \frac{\cos. \frac{1}{2} (p - q)}{\sin. \frac{1}{2} (p - q)},$$

hence,

$$\frac{\sin. p + \sin. q}{\sin. p - \sin. q} = \frac{\sin. \frac{1}{2} (p + q)}{\cos. \frac{1}{2} (p + q)} \div \frac{\sin. \frac{1}{2} (p - q)}{\cos. \frac{1}{2} (p - q)} = \frac{\text{tang. } \frac{1}{2} (p + q)}{\text{tang. } \frac{1}{2} (p - q)}. \quad (9.)$$

Numerical Values of Sines, Tangents, &c.

§ 17. Having deduced a few of the most simple relations of the trigonometrical functions of angles, we will now proceed to determine their numerical values.

Sines of very small angles are obviously very nearly equal to the arcs which measure these angles. We have found (§ 6) that the length of an arc of 1' is equal to 0.0002908882, &c. If we regard this as the sine of 1', from which, as will be hereafter shown, it does not differ even in its eleventh decimal figure, we may find the cosine of 1' as follows (§ 8) :

$$\cos. 1' = \sqrt{1 - \sin.^2 1'} = 0.9999999577, \text{ \&c.}$$

Having thus found the sine and cosine of 1', we may continue our work for larger angles by the aid of equations (1) and (3), (§16), which give

$$\begin{aligned} \sin. (a + b) &= 2 \sin. a \cos. b - \sin. (a - b). \\ \cos. (a + b) &= 2 \cos. a \cos. b - \cos. (a - b). \end{aligned}$$

Putting $b=1'$ and $a=1', 2', 3', 4', \text{ \&c.}$, successively, we shall obtain

$$\begin{aligned} \sin. 2' &= 2 \sin. 1' \cos. 1' - \sin. 0'. \\ \sin. 3' &= 2 \sin. 2' \cos. 1' - \sin. 1'. \\ \sin. 4' &= 2 \sin. 3' \cos. 1' - \sin. 2'. \\ \sin. 5' &= 2 \sin. 4' \cos. 1' - \sin. 3'. \\ \&c. \qquad \&c. \qquad \&c. \end{aligned}$$

$$\begin{aligned} \cos. 2' &= 2 \cos. 1' \cos. 1' - \cos. 0'. \\ \cos. 3' &= 2 \cos. 2' \cos. 1' - \cos. 1'. \\ \cos. 4' &= 2 \cos. 3' \cos. 1' - \cos. 2'. \\ \cos. 5' &= 2 \cos. 4' \cos. 1' - \cos. 3'. \\ \&c. \qquad \&c. \qquad \&c. \end{aligned}$$

Using the above value of sine and cosine of $1'$, executing the work above indicated, and preserving only nine decimal places, we find

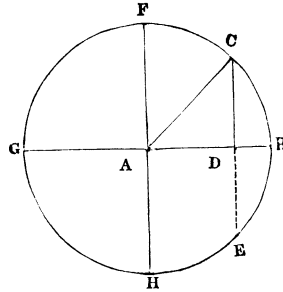
sin. $1' = 0.000290888$.	cos. $1' = 0.999999958$.
sin. $2' = 0.000581776$.	cos. $2' = 0.999999831$.
sin. $3' = 0.000872665$.	cos. $3' = 0.999999619$.
sin. $4' = 0.001163553$.	cos. $4' = 0.999999323$.
sin. $5' = 0.001454441$.	cos. $5' = 0.999998942$.
&c.	&c.

The work might be continued in this way to any desired extent.

§ 18. We will now seek formulas which shall give the sine and cosine of any arc without our being obliged first to compute those of a smaller arc.

Denote the arc BC, which measures the angle BAC, by x , the radius being unity.

Then assume,



$$CD = \sin. x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 +, \&c. \quad (1.)$$

$$DA = \cos. x = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 +, \&c. \quad (2.)$$

When $x = 0$, the sine of x becomes zero, which requires a_0 to $= 0$. Since the sine of a negative arc (§ 12) is the same as *minus* the sine of the same arc taken positively, it follows that the expression (1) for the sine of x must be of such a form as to change the algebraic signs of all its terms when $-x$ is written for $+x$, consequently it cannot contain any even powers of x . Therefore $a_2 = 0$; $a_4 = 0$, &c.

Again, when $x = 0$, the cosine of x becomes 1, so that $b_0 = 1$. And, since the cosine of a negative arc is precisely the same as the cosine of the same arc taken positively, it follows that the expression for the cosine of x must be of such a form as not to change the algebraic signs of any of its terms when $-x$ is written for $+x$, consequently it cannot contain any odd powers of x . Therefore $b_1 = 0$; $b_3 = 0$; $b_5 = 0$, &c.

Equations (1) and (2) must therefore assume the following forms :

$$\sin. x = a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 +, \&c. \quad . \quad . \quad (3.)$$

$$\cos. x = 1 + b_2 x^2 + b_4 x^4 + b_6 x^6 +, \&c. \quad . \quad . \quad (4.)$$

If we suppose the arc x to be exceedingly small, we may obtain a very close approximate value for the sine of x , in (3); by retaining only the first power of x , we thus find

$$\sin. x = a_1 x, \text{ very nearly.}$$

But in very small arcs the sine and the arc are very nearly equal; hence, when x is very small, we must have $\sin. x = x$, very nearly. Consequently $a_1 = 1$.

Using this value of a_1 , equations (3) and (4) become

$$\sin. x = x + a_3 x^3 + a_5 x^5 + a_7 x^7 +, \&c. \quad . \quad . \quad (5.)$$

$$\cos. x = 1 + b_2 x^2 + b_4 x^4 + b_6 x^6 +, \&c. \quad . \quad . \quad (6.)$$

It now remains to find the other coefficients, $a_3, a_5, a_7, \&c., b_2, b_4, b_6, \&c.$ For this purpose we proceed as follows :

In (5) write $2x$ for x , and we obtain

$$\sin. 2x = 2x + 8a_3 x^3 + 32a_5 x^5 + 128a_7 x^7 +, \&c. \quad (7.)$$

Dividing (7) by 2, we have

$$\frac{1}{2} \sin. 2x = x + 4a_3 x^3 + 16a_5 x^5 + 64a_7 x^7 +, \&c. \quad (8.)$$

Taking the product of (5) and (6), we find

$$\sin. x \cos. x = x + a_3 \left\{ \begin{array}{l} x^3 + \\ + b_2 \end{array} \right\} + a_5 \left\{ \begin{array}{l} x^5 + b_2 a_3 \\ + b_4 \end{array} \right\} + a_7 \left\{ \begin{array}{l} x^7 + b_2 a_5 \\ + b_4 a_3 \\ + b_6 \end{array} \right\} +, \&c. \quad (9.)$$

The left-hand members of (8) and (9) being equal [§ 15, eq. (5)], their right-hand members must also be equal; hence equating the coefficients of like powers of x , we obtain

$$\left. \begin{array}{l} 4a_3 = a_3 + b_2 \\ 16a_5 = a_5 + b_2 a_3 + b_4 \\ 64a_7 = a_7 + b_2 a_5 + b_4 a_3 + b_6 \\ \&c. \qquad \qquad \&c. \end{array} \right\} . \quad . \quad . \quad (10.)$$

Equations (5) and (6), when squared, become respectively

$$\sin.^2 x = x^2 + 2 a_3 x^4 + \frac{a_3^2}{+ 2 a_5} \left\{ x^6 +, \&c. \right. \quad (11.)$$

$$\cos.^2 x = 1 + 2 b_2 x^2 + \frac{b_2^2}{+ 2 b_4} \left\{ x^4 + 2 b_6 \right. \left. + 2 b_2 b_4 \right\} x^6 +, \&c. \quad (12.)$$

If from the sum of (11) and (12) we subtract 1, we shall obtain

$$\sin.^2 x + \cos.^2 x - 1 = + 1 \left\{ \begin{array}{l} x^2 + 2 a_3 \\ + 2 b_2 \end{array} \right\} \left\{ \begin{array}{l} x^4 + 2 a_5 \\ + 2 b_4 \end{array} \right\} \left\{ \begin{array}{l} + a_3^2 \\ x^6 + 2 a_5 \\ + 2 b_6 \\ + 2 b_2 b_4 \end{array} \right\} x^6 +, \&c. \quad (13.)$$

Since the left-hand member of (13) is equal to zero [§ 8, eq. 5 of B], the right-hand member must also equal zero; hence the coefficients of the different powers of x must equal zero, which leads to the following conditions:

$$\left. \begin{array}{l} 1 + 2 b_2 = 0 \\ 2 a_3 + b_2^2 + 2 b_4 = 0 \\ a_3^2 + 2 a_5 + 2 b_6 + 2 b_2 b_4 = 0 \\ \&c. \qquad \qquad \&c. \end{array} \right\} \quad (14.)$$

Equations (10) and (14) give immediately these results:

$$\begin{aligned} b_2 &= -\frac{1}{2} = -\frac{1}{1 \times 2}. \\ a_3 &= -\frac{1}{6} = -\frac{1}{1 \times 2 \times 3}. \\ b_4 &= \frac{1}{24} = \frac{1}{1 \times 2 \times 3 \times 4}. \\ a_5 &= \frac{1}{120} = \frac{1}{1 \times 2 \times 3 \times 4 \times 5}. \\ b_6 &= -\frac{1}{720} = -\frac{1}{1 \times 2 \times 3 \times 4 \times 5 \times 6}. \\ a_7 &= -\frac{1}{5040} = -\frac{1}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7}. \\ &\&c. \qquad \&c. \qquad \&c. \end{aligned}$$

Hence, equations (5) and (6) become

$$\sin. x = x - \frac{x^3}{1 \times 2 \times 3} + \frac{x^5}{1 \times 2 \times 3 \times 4 \times 5} - \frac{x^7}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7} +, \&c. \quad (15.)$$

$$\cos. x = 1 - \frac{x^2}{1 \times 2} + \frac{x^4}{1 \times 2 \times 3 \times 4} - \frac{x^6}{1 \times 2 \times 3 \times 4 \times 5 \times 6} +, \&c. \quad (16.)$$

If we divide the right-hand member of (15) by the right-hand member of (16), we shall obtain an infinite series, giving the value of $\text{tang. } x$, as follows :

$$\text{tang. } x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} +, \&c.$$

Other series might be found by combining (15) and (16) for the $\text{cotan. } x$, $\text{sec. } x$, and $\text{cosec. } x$.

The arc of $1'$ has already been found [§ 6] to be equal to 0.0002908882086, &c. Putting this value for x , in the above series, we find

$$\frac{x^2}{1 \times 2} = 0.0000000423079,$$

$$\frac{x^3}{1 \times 2 \times 3} = 0.0000000000041.$$

If we confine ourselves to thirteen decimals, we shall obtain

$$\sin. 1' = x - \frac{x^3}{1 \times 2 \times 3} = 0.0002908882045 ;$$

$$\cos. 1' = 1 - \frac{x^2}{1 \times 2} = 0.9999999576921.$$

From this we see that the sine of $1'$ does not differ from its corresponding arc until after we pass the 11th decimal place, as was asserted under § 17.

For ordinary purposes, seven decimal places is as accurate as is desired. Having already (§ 17) computed the sine and cosine of each minute as far as $5'$, we will now, by the aid of equations (15) and (16), continue the computation, limiting ourselves to seven decimals, and shall obtain as follows :

arc of $6'$	$= 0.0017453,$	$\sin. = 0.0017453,$	$\cos. = 0.9999985.$
“ $7'$	$= 0.0020362,$	“ $= 0.0020362,$	“ $= 0.9999979.$
“ $8'$	$= 0.0023271,$	“ $= 0.0023271,$	“ $= 0.9999973.$
“ $9'$	$= 0.0026180,$	“ $= 0.0026180,$	“ $= 0.9999966.$
“ $10'$	$= 0.0029089,$	“ $= 0.0029089,$	“ $= 0.9999958.$
“ $11'$	$= 0.0031998,$	“ $= 0.0031998,$	“ $= 0.9999949.$
“ $12'$	$= 0.0034907,$	“ $= 0.0034907,$	“ $= 0.9999939.$
“ $13'$	$= 0.0037815,$	“ $= 0.0037815,$	“ $= 0.9999928.$
“ $14'$	$= 0.0040724,$	“ $= 0.0040724,$	“ $= 0.9999917.$
“ $15'$	$= 0.0043633,$	“ $= 0.0043633,$	“ $= 0.9999905.$

In the same way, for degrees, we find

arc of $1^\circ = 0.0174533$,	$\sin. = 0.0174524$,	$\cos. = 0.9998477$.
" $2^\circ = 0.0349066$,	" $= 0.0348995$,	" $= 0.9993908$.
" $3^\circ = 0.0523599$,	" $= 0.0523360$,	" $= 0.9986295$.
" $4^\circ = 0.0698132$,	" $= 0.0697565$,	" $= 0.9975641$.
" $5^\circ = 0.0872665$,	" $= 0.0871557$,	" $= 0.9961947$.
" $6^\circ = 0.1047198$,	" $= 0.1045285$,	" $= 0.9945219$.
" $7^\circ = 0.1221730$,	" $= 0.1218693$,	" $= 0.9925462$.
" $8^\circ = 0.1396263$,	" $= 0.1391731$,	" $= 0.9902681$.
" $9^\circ = 0.1570796$,	" $= 0.1564345$,	" $= 0.9876883$.
" $10^\circ = 0.1745329$,	" $= 0.1736482$,	" $= 0.9848078$.

Having in this way computed the sines and cosines, we may [see equations (B), § 8] obtain the tangents by dividing the sines by their corresponding cosines; the cotangents will be obtained by dividing the cosines by the sines. The reciprocal of the cosines will give the secants, and the reciprocal of the sines will give the cosecants.

The values of the sines, cosines, tangents, and cotangents are given in Table III.

CHAPTER II.

EXPLANATION OF TABLES.

TABLE I.

LOGARITHMS.

§ 19. THIS table gives the logarithms of all numbers from 1 to 10000.

For the method of calculating these logarithms, the student is referred to my "Treatise on Algebra," chap. IX.

The common logarithm, or *Briggean* logarithm, as sometimes called, in honor of the inventor *Briggs*, has 10 for its base. Hence, for common logarithms, we have

$10^0=1$	, that is, 0 is the logarithm of 1
$10^1=10$	, “ “ 1 “ “ “ “ 10
$10^2=100$	, “ “ 2 “ “ “ “ 100
$10^3=1000$	, “ “ 3 “ “ “ “ 1000
$10^4=10000$	, “ “ 4 “ “ “ “ 10000
$10^5=100000$,	“ “ 5 “ “ “ “ 100000
&c.	&c. &c.

If we use negative exponents, we shall have

$10^{-1}=0\cdot1$	, that is, -1 is the logarithm of $0\cdot1$
$10^{-2}=0\cdot01$	, “ “ -2 “ “ “ “ $0\cdot01$
$10^{-3}=0\cdot001$	, “ “ -3 “ “ “ “ $0\cdot001$
$10^{-4}=0\cdot0001$	, “ “ -4 “ “ “ “ $0\cdot0001$
$10^{-5}=0\cdot00001$,	“ “ -5 “ “ “ “ $0\cdot00001$
&c.	&c. &c.

From the above, we see that the logarithm of any number between 1 and 10 is a proper fraction between 0 and 1. The logarithm of any number between 10 and 100 is 1 plus a proper fraction. The logarithm of any number between 100 and 1000 is 2 plus a fraction.

We also see that the logarithm of any number between 1 and $0\cdot1$ is some number between 0 and -1 ; it may therefore be represented by -1 plus a fraction. For a similar reason, the logarithm of any number between $0\cdot1$ and $0\cdot01$ may be represented by -2 plus a fraction. The logarithm of any number between $0\cdot01$ and $0\cdot001$ is -3 plus a fraction.

Hence, we see that logarithms of all numbers greater than 10 or less than 1 consist of an integral part, positive or negative, plus a proper fraction. The integral part is called the *characteristic*, and may readily be assigned by the following

RULE.

The characteristic of the logarithm of any number greater than 1 is one less than the number of places of figures which express the integral part of the given number.

The characteristic of the logarithm of a decimal fraction is a negative number, and is equal to the number of places by which the first significant figure is removed from the units' place.

Thus, we have as follows :

<i>Number.</i>	<i>Characteristic.</i>	<i>Number.</i>	<i>Characteristic.</i>
4	0	0.4	-1
48	1	0.46	-1
138	2	0.375	-1
2304	3	0.05	-2
4.7	0	0.0578	-2
37.04	1	0.006	-3
120.005	2	0.00675	-3
2178.88	3	0.00035	-4

The characteristic being so readily obtained by means of the above rule, has been omitted in the tables, except in the small table on page 1, where are found the logarithms of all numbers from 1 to 100, with their characteristics. But for all the other numbers of our table only the decimal part of the logarithms are given.

The decimal part of all logarithms are regarded as positive. The characteristic may, however, be considered as negative. When this is the case, it is usual to place the negative sign immediately over the characteristic. Thus the logarithm of 0.75 is $\bar{1}.875061$, or $0.875061-1$.

§ 20. *To find, in the table, the logarithm of any number from 1 to 100.*

Seek for the given number in one of the columns headed N, of the first page of the tables of logarithms, and against it on the right in the next column, will be found the logarithm. We thus find

$$\begin{aligned}\log. 7 &= 0.845098, \\ \log. 39 &= 1.591065.\end{aligned}$$

§ 21. *To find the logarithm of any number consisting of three places of figures.*

Seek the given number in one of the left-hand columns headed N, and against it in the column headed 0 will be found the decimal part of the logarithm, observing that if only four figures

are given in column 0, they are the last four : the first two are to be taken from the number directly above which contains six places of figures. Thus, we find

$$\log. 617 = 2.790285,$$

$$\log. 635 = 2.802774.$$

§ 22. *To find the logarithm of a number consisting of four places of figures.*

Seek the first three figures in column headed N, then passing horizontally across the page until you reach the column headed with the fourth figure of the given number, and you will have the last four decimal figures of the logarithm. The first two figures are to be taken from the column headed 0 ; observing, if the four figures found, stand opposite to a row of six figures in the column 0, to use the first two. But if the four figures found are opposite a line of only four figures, you are then to ascend the column till you come to a line of six figures, and then to use the two left-hand figures. The six figures thus obtained will be the decimal part of the logarithm. The characteristic is to be found by the Rule under § 19. Thus, we find

$$\log. 3657 = 3.563125,$$

$$\log. 7641 = 3.883150.$$

In some portions of the table, in the columns headed 1, 2, 3, 4, &c., the character * is to be found in place of a figure, which indicates that the two figures of the first column, which are to be prefixed, have changed, and are to be taken from the horizontal line directly below. This change of the two left-hand figures of the logarithm is indicated by an asterisk placed in the column 0. The place occupied by this character is to be supplied with a zero. The two figures from the column 0 must also be taken from the line below the asterisk, if the character * has been passed over while crossing the page horizontally. Thus, we find

$$\log. 2138 = 3.330008,$$

$$\log. 2298 = 3.361350,$$

$$\log. 3237 = 3.510143.$$

§ 23. *To find the logarithm of a number consisting of five or more places of figures.*

For example, let us seek the logarithm of 734582. We readily find as follows :

$$\begin{array}{rcl} \log. 734500 & = & 5.865992 \\ \log. 734600 & = & 5.866051 \\ \hline \text{Diff. of numb.} = 100 & & \text{Diff. of logs.} = 59 \end{array}$$

The difference between the first number 734500 and the given number is $734582 - 734500 = 82$. Hence, if we suppose the difference of logarithms to be to each other as the difference of their corresponding numbers, which supposition is very nearly correct, we shall have

$$100 : 82 :: 59 : 48.38,$$

for the difference between the logarithm of 734500 and the logarithm of 734582. Hence, we have

$$\begin{array}{r} \log. 734500 = 5.865992 \\ \quad \quad \quad + 48.38 \\ \hline \log. 734582 = 5.866040 \end{array}$$

Had there been only five figures in the given number, that is, only one beyond the fourth place, the first term of the above proportion would have been 10 instead of 100; had there been three additional figures beyond the fourth place, the first term would have been 1000.

The difference between the logarithms of consecutive numbers of four places of figures is given in the table in a column headed D. Thus by turning to the table, we find 59, our third term of the above proportion, immediately opposite the logarithm of 7345.

From what has been done, we see that we may find the logarithm of a number consisting of more than four places of figures by the following method :

Consider all the figures after the fourth as zeros. Then find the decimal part of the logarithm of the number given by the first four figures, observing to give a characteristic for the whole number of figures by rule under § 19. Take from the column

On the number which is found directly opposite the logarithm already taken out, and multiply it by the figures which were regarded as zeros, pointing off in the product as though these figures were all decimals; add the result thus obtained to the logarithm already found, and it will give the logarithm of the given number.

In this way we find

$$\log. 365365 = 5.562727.$$

$$\log. 635536 = 5.803140.$$

$$\log. 704307 = 5.847762.$$

NOTE.—Since the foregoing process of finding the logarithm of a number of more than four places of figures is founded on the supposition that the differences of logarithms of different numbers are to each other as the differences of the numbers, which supposition is not strictly true, it follows that this method can be used only to a limited extent. It ought never to be employed for a number consisting of more than six places of figures.

§ 24. To find the logarithm of a Vulgar Fraction.

Since a vulgar fraction is the quotient of the numerator divided by the denominator, we may obtain its logarithm by subtracting the logarithm of the denominator from the logarithm of the numerator. Thus, the logarithm of $\frac{3}{5}$ is $\log. 3 - \log. 5 = 1.568202 - 1.724276 = \bar{1}.843926$.

In a similar manner we find

$$\log. \frac{3}{4} = \log. 3 - \log. 4 = 1.568202 - 1.602060 = \bar{1}.966142.$$

$$\log. \frac{3}{5} = \log. 3 - \log. 5 = 1.568202 - 1.724276 = \bar{1}.843926.$$

§ 25. To find the logarithm of a Decimal Fraction.

Since we have, by the property of logarithms,

$$\log. 4.5 = \log. \frac{45}{10} = \log. 45 - \log. 10 = \log. 45 - 1;$$

$$\log. 3.65 = \log. \frac{365}{100} = \log. 365 - \log. 100 = \log. 365 - 2;$$

$$\log. 3.754 = \log. \frac{3754}{1000} = \log. 3754 - \log. 1000 = \log. 3754 - 3;$$

it follows that the decimal part of the logarithm of a decimal fraction is the same as though the number was wholly integral, the only difference between the logarithm of a decimal number and of the number considered as integral is in the characteristic. Hence, take out the logarithm as though the number were integral, and fix a characteristic according to rule under § 19.

We thus find,

$\log. 476$	$= 2.677607$
$\log. 47.6$	$= 1.677607$
$\log. 4.76$	$= 0.677607$
$\log. 0.476$	$= \bar{1}.677607$
$\log. 0.0476$	$= \bar{2}.677607$
$\log. 0.00476$	$= \bar{3}.677607$

§ 26. *To find the natural number corresponding to any logarithm.*

Seek in the table, in the column 0, for the first two figures of the decimal part of the logarithm; the other four figures are to be sought for in the same column, or in any one of the columns 1, 2, 3, &c. If the decimal part of the logarithm is exactly found, then will the first three figures of the corresponding number be found in the column N, and the fourth figure will be found at the top of the page. This number must be made to correspond with the given characteristic of the given logarithm by annexing ciphers, or by pointing off decimals. Thus the

logarithm 5.311754	gives	205000,
“ 4.341237	“	21940,
“ 3.759970	“	5754,
“ 2.759970	“	575.4,
“ 1.759970	“	57.54.

When the decimal part of the logarithm cannot be accurately found in the table, take out the four figures corresponding to the next less logarithm. Then for the additional figures, subtract this less logarithm from the given logarithm, and divide the remainder, with naughts annexed, by the corresponding number taken from column D. For example, let us seek the number whose logarithm is 1.234567. We find the next less number to the decimal 0.234567 to be 0.234517, which corresponds to 1716. We also find the number in column D to be 253. Hence

$$\begin{array}{r}
 0.234567 \\
 0.234517 \\
 \hline
 0.000050
 \end{array}
 \qquad
 \begin{array}{r}
 0.234770 \\
 0.234517 \\
 \hline
 253
 \end{array}$$

50; and $50 \div 253 = 0.198$, nearly.

So that the number answering to the logarithm 1.234567 is 17.16198, nearly.

ARITHMETICAL CALCULATIONS BY LOGARITHMS.

§ 27. *Multiplication by Logarithms.*

Since the logarithm of the product of two or more factors is equal to the sum of their logarithms, we deduce, for multiplication by logarithms, this

RULE.

Add the logarithms of the factors, and the sum will be the logarithm of the product.

1. What is the product of 3.65 by 56.3?

$$\log. 3.65 = 0.562293$$

$$\log. 56.3 = 1.750508$$

$$\log. \text{ of product} = \underline{2.312801},$$

which gives 205.495 for the product.

2. What is the product of 7.8 by 35.3? *Ans.* 275.34

3. What is the product of 2.13 by 0.57? *Ans.* 1.2141.

NOTE.—When any of the characteristics of the logarithms are negative, we must observe the *algebraic rule* for their addition.

4. What is the continued product of 53.7, 0.12, and 0.004?

$$\log. 53.7 = 1.729974$$

$$\log. 0.12 = \bar{1}.079181$$

$$\log. 0.004 = \bar{3}.602060$$

$$\text{product} = 0.025776, \text{ whose } \log. = \underline{\bar{2}.411215}$$

5. What is the square of 3.7; that is, what is the product of 3.7 by 3.7? *Ans.* 13.69.

6. What is the cube of 3.8; that is, what is the continued product of 3.8, 3.8, and 3.8? *Ans.* 54.872.

§ 28. *For Division by Logarithms*, we obviously have this

RULE.

Subtract the logarithm of the divisor from the logarithm of the dividend.

EXAMPLES.

1. What is the quotient of 365 by 7·3?

$$\log. 365 = 2\cdot562293$$

$$\log. 7\cdot3 = 0\cdot863323$$

$$\text{quotient is } 5, \text{ its log.} = 1\cdot698970$$

2. What is the quotient of 2·456 by 1·47?

Ans. 1·67075, nearly.

3. What is the quotient of 7·4 by 12·58?
- Ans.*
- 0·588235.

NOTE.—When either or both of the characteristics of the logarithms are negative, we must observe the *algebraic rule* for the subtraction of the one from the other.

4. What is the quotient of 0·378 divided by 0·45?

$$\log. 0\cdot378 = \bar{1}\cdot577492$$

$$\log. 0\cdot45 = \bar{1}\cdot653213$$

$$\text{quotient} = 0\cdot84, \text{ whose log.} = \bar{1}\cdot924279$$

5. What is the quotient of 0·10071 by 0·00373?

$$\log. 0\cdot10071 = \bar{1}\cdot003072$$

$$\log. 0\cdot00373 = \bar{3}\cdot571709$$

$$\text{quotient} = 27, \text{ whose log.} = 1\cdot431363, \text{ nearly.}$$

§ 29. *Involution by Logarithms.*

Since the exponent denoting any power of any number expresses how many times this number is used as a factor to produce the given power, it follows that the logarithm of any power is equal to the logarithm of the number repeated as many times as there are units in the exponent. Hence we have this

RULE.

Multiply the logarithm of the number by the exponent denoting the power.

EXAMPLES.

1. What is the 5th power of 1·234?

$$\log. 1\cdot234 = 0\cdot091315$$

5 multiply.

$$\text{power} = 2\cdot86137, \text{ whose log.} = 0\cdot456575$$

2. What is the 7th power of 0.5?

$$\log. 0.5 = \bar{1}.698970$$

7 multiply.

$$\text{power} = 0.0078125, \text{ whose } \log. = \bar{3}.892790$$

NOTE.—It must be kept in mind that the decimal part of every logarithm is positive, so that, as in the last example, whatever is to carry to the product of the characteristic by the exponent is positive.

3. What is the 30th power of 1.07? *Ans.* 7.6123, nearly.

4. What is the 11th power of 0.11?

Ans. 0.0000000000285313.

§ 30. *Evolution by Logarithms.*

Since the exponent denoting a root indicates that the number is to be separated into as many equal factors as there are units in the exponents, we obviously have this

RULE.

Divide the logarithm of the number by the number denoting the root.

EXAMPLES.

1. What is the 11th root of 11?

$\log. 11 = 1.041393$, which divided by 11 gives 0.094672 for the log. of the root. Hence the root = 1.24357, nearly.

NOTE.—When the characteristic is negative, and not divisible by the exponent, we must put with it a sufficiently large negative integer to make it divisible, and then connect with the decimal portion of the logarithm an equally large positive number. This will be best illustrated by the following example.

2. What is the 5th root of 0.00567?

$\log. 0.00567 = \bar{3}.753583 = \bar{5} + 2.753583$, which divided by 5 gives $\bar{1}.550716$ for the logarithm of the root. Hence the root = 0.3554, nearly.

3. What is the 3d root of 0.365? *Ans.* 0.714657, nearly.

4. What is the 7th root of 7? *Ans.* 1.32047, nearly.

5. What is the 5th root of 0.5? *Ans.* 0.87055, nearly.

§ 31. *Proportion by Logarithms.*

Since the fourth term of a proportion is found by dividing the product of the second and third terms by the first, we obviously have this

RULE.

From the sum of the logarithms of the second and third terms, subtract the logarithm of the first term.

What is the fourth term of a proportion of which the first, second, and third terms are respectively 0·0146, 4·5, and 1·07?

$$\begin{array}{r}
 \log. 4\cdot5 = 0\cdot653213 \\
 \log. 1\cdot07 = 0\cdot029384 \\
 \hline
 0\cdot682597 \\
 \text{subtract } \log. 0\cdot0146 = \underline{2\cdot164353} \\
 \log. \text{ of fourth term} = 2\cdot518244 \\
 \text{Consequently the fourth term} = 329\cdot794, \text{ nearly.}
 \end{array}$$

ARITHMETICAL COMPLEMENT.

§ 32. The subtraction of a logarithm from the sum of two or more logarithms is usually made to depend upon addition, by making use of its *Arithmetical Complement*, which is the difference between 10 and the given logarithm, and is readily taken from the table, by beginning at the left hand and subtracting each figure from 9, except the last significant figure on the right, which must be taken from 10. Now it is obvious, that if instead of subtracting a given number from another, we first subtract the number from 10, and then add the result, we shall, after rejecting 10, obtain the true difference. Hence to work a proportion by logarithms, we may use this second

RULE.

Take the arithmetical complement of the logarithm of the first term, and the logarithms of the second and third terms, and from the sum of the three reject 10 from the characteristic.

EXAMPLES.

1. What is the fourth term of the proportion $6.5 : 30.5 :: 1.25 ?$

$$\text{Arith. comp. log. } 6.5 = 9.187087$$

$$\text{log. } 30.5 = 1.484300$$

$$\text{log. } 1.25 = 0.096910$$

sum, after rejecting 10, = 0.768297, which
is the log. of 5.86539, the fourth term.

2. What is the fourth term in the proportion $3456 : 10.5 :: 6543 ?$

$$\text{ar. co. log. } 3456 = 6.461426$$

$$\text{log. } 10.5 = 1.021189$$

$$\text{log. } 6543 = 3.815777$$

$$\text{fourth term} = 19.8788, \text{ log.} = 1.298392$$

3. In the proportion $1.23 : 2.34 :: 3.45$, what is the fourth term?
Ans. 6.56342, nearly.

TABLE III.

NATURAL SINES AND TANGENTS.

§ 33. It is necessary to explain the arrangement of this table before explaining that of Table II., since the values of the latter have been obtained from those of the former. Table II., being logarithmic values, is with propriety placed immediately after the table of logarithms.

Table III. contains the natural sines and tangents to every minute of the first quadrant, and consequently those of the co-sines and cotangents. The first 9 pages are devoted to sines and cosines, the remainder of this table gives the tangents and cotangents.

When the number of degrees does not exceed 45, they are to be found at the top of the page, and the additional minutes are given in the first or left-hand column of the page. But if the degrees exceed 45, they are to be found at the bottom of the page, and the additional minutes are then given in the last, or right-hand column of the page.

It will be seen by examining this table that the number of degrees at the top of any page added to the number at the bottom of the same, gives 89, to which adding the 60 intermediate minutes makes just 90 degrees. Thus, on page 70 we have 34° at the top and 55° at the bottom, the sum of which is 89° .

Since the cosine and cotangent (§ 8) of an angle is the same as the sine and tangent respectively of its complement, it follows that by this arrangement of the table those columns which are marked sine and tangent at the top of the page will be marked cosine and cotangent respectively, at the bottom of the page.

As the sine and cosine of any angle cannot exceed the radius, which is a unit, it follows that their values are always expressed in this table by a decimal. The tangent and cotangent may in some cases exceed a unit, and will then be expressed by a decimal joined to an integral part.

Thus, turning to this table, we find

$$\begin{aligned} \sin. 17^\circ 13' &= 0.29599; \cos. 17^\circ 13' = 0.95519. \\ \text{tang. } 17^\circ 13' &= 0.30987; \text{cotan. } 17^\circ 13' = 3.22715. \\ \sin. 55^\circ 43' &= 0.82626; \cos. 55^\circ 43' = 0.56329. \\ \text{tang. } 55^\circ 43' &= 1.46686; \text{cotan. } 55^\circ 43' = 0.68173. \end{aligned}$$

TABLE II.

LOGARITHMIC SINES AND TANGENTS.

§ 34. This table is constructed by taking the logarithms of the values given in Table III., and increasing each logarithm by 10. The arrangement is the same as given in the Table of Natural Sines, Tangents, &c.; that is, the degrees are to be found at the top of the page and minutes in the first column, when the angle is less than 45° . When the angle is greater than 45° , the degrees are to be found at the bottom of the page and the minutes in the last column. As in the last table, the column marked sine at top is cosine at the bottom; the column marked tangent at the top is cotangent at the bottom, and conversely.

Obtuse angles are also provided for in this table: when found at the top of the page they are at the right-hand side, and the

minutes are then to be found in the right-hand column. But when found at the bottom of the page they are at the left-hand side, and the minutes are then to be found in the left-hand column.

As most cases of Trigonometry are solved by the assistance of proportions, which are most readily wrought by the help of logarithms, it has been found convenient to prepare beforehand the logarithms of the sines and tangents of all angles expressed by degrees and minutes within the limit of the first quadrant.

Since the sines, as well as tangents, of many arcs are less than a unit, the characteristic of their logarithms will be negative. By adding 10 to each logarithm, the characteristics will all be positive, and we shall not be as liable to commit errors in using them, as we should if some had positive characteristics, and others negative characteristics.

§ 35. *To find, in the table, the logarithmic sine, cosine, tangent, or cotangent of any given angle.*

If the angle is less than 45° , we seek the degrees at the top of the page and the minutes in the first column on the left; then passing across the page horizontally until we come to the column having the appropriate heading of sine, cosine, tangent, or cotangent, as the case may require, we shall find the logarithm sought. Thus, on page 53, we find

$$\begin{aligned}\sin. \quad 35^\circ 53' &= 9.767999; \quad \cos. \quad 35^\circ 53' = 9.908599. \\ \text{tang. } 35^\circ 53' &= 9.859400; \quad \text{cotan. } 35^\circ 53' = 10.140600.\end{aligned}$$

If the angle is greater than 45° , we seek for the degrees at the bottom of the page, and the minutes on the right-hand side of the page; then, as before, passing horizontally across the page till we reach the column having at the bottom the appropriate designation of sine, cosine, tangent, or cotangent, as the case may require, and the number thus given will be the one sought. If the angle is obtuse, we must make use of its supplement (§ 13).

§ 36. When the angle is expressed in degrees, minutes, and seconds, we proceed as follows:

Let it be required, for example, to find the logarithmic

sine of $13^{\circ} 14' 17''$. By what has already been said, we readily find

$$\sin. 13^{\circ} 14' = 9.359678$$

$$\sin. 13^{\circ} 15' = 9.360215$$

$$537 = \text{diff. for } 1'.$$

Hence, $537 \div 60 = 8.95 = \text{diff. for } 1''$, and $895 = \text{diff. for } 100''$. If we suppose the logarithmic sines to increase in the same ratio as the increase of their corresponding arcs, which supposition is very nearly correct, we shall be able to find the difference corresponding to $17''$ by multiplying 8.95 by 17 , which gives 152.15 .

$$\sin. 13^{\circ} 14' = 9.359678$$

$$152 \text{ add.}$$

$$9.359830 = \text{sine of } 13^{\circ} 14' 17''.$$

The column D, immediately at the right of the column of sines, gives the differences of the logarithmic sines for $100''$. Thus, in the tables opposite the sine of $13^{\circ} 14'$ in the column D, we find 895 , which we have already shown to be the increase for $100''$. In the same way, the column D, immediately after the column of cosines, gives the difference of the logarithmic cosines for $100''$. The column D, between the tangent and cotangent columns, answers for both, giving also the difference for $100''$.

Hence, to find the logarithmic sine of an arc given in degrees, minutes, and seconds, we must first seek the value corresponding to the degrees and minutes; then multiply the corresponding tabular number in column D by the number of seconds, cutting off two places to the right for decimals; add the product to the value already found, and the result will be the logarithmic sine sought. Thus, find the sine of $37^{\circ} 31' 13''$.

Sin. $37^{\circ} 31' = 9.784612$. Tabular number D = 274 , which multiplied by 13 , the number of seconds, gives $274 \times 13 = 3562$; pointing off two decimals, we have 35.62 , or 36 nearly, which added gives

$$\sin. 37^{\circ} 31' = 9.784612$$

$$36$$

$$\sin. 37^{\circ} 31' 13'' = 9.784648$$

The tangent of an arc where there are seconds is found by the same method as has been employed for finding the sine. In finding the values of the cosine and cotangent, we must bear in mind that the cosines and cotangents decrease as the arcs increase, consequently the corrections found by aid of columns D must be subtracted instead of being added.

Find the cosine of $23^{\circ} 6' 47''$.

$$\begin{array}{rcl} \cos. 23^{\circ} 6' & = & 9.963704 \\ & \text{subtract } 42.30 = \frac{4.7}{1000} \text{ of } 90 \text{ (D.)} & \\ \hline \cos. 23^{\circ} 6' 47'' & = & 9.963662 \end{array}$$

Find tangent and cotangent of $30^{\circ} 30' 30''$.

$$\begin{array}{rcl} \text{tang. } 30^{\circ} 30' & = & 9.770148 \\ & \text{add } 144 = \frac{3.0}{1000} \text{ of } 481 \text{ (D.)} & \\ \hline \text{tang. } 30^{\circ} 30' 30'' & = & 9.770292 \\ \\ \text{cotan. } 30^{\circ} 30' & = & 10.229852 \\ & \text{subtract } 144 = \frac{3.0}{1000} \text{ of } 481 \text{ (D.)} & \\ \hline \text{cotan. } 30^{\circ} 30' 30'' & = & 10.229708 \end{array}$$

§ 37. *To find the arc expressed in degrees, minutes, and seconds, when its logarithmic sine, cosine, tangent, or cotangent is given.*

Seek for the given value in the column having the proper designation of sine, cosine, tangent, or cotangent, as the case may require, keeping in mind (§ 34) that the columns designated by sine and cosine at the top of the page are marked cosine and sine respectively at the bottom of the page. Also, those marked tangent and cotangent at the top are respectively cotangent and tangent at the bottom.

If the exact value is found in the table, the arc will be accurately given in degrees and minutes. The degrees are to be taken from the top of the page, and the minutes from the left-hand column, when the proper precept of the column is found at the top. But when the precept is found at the bottom, the degrees are to be taken from the bottom of the page and the minutes from the right-hand column.

If the number cannot be exactly found in the table, then find, as above, the degrees and minutes corresponding to the nearest

less logarithm. Divide the difference of these logarithms, with two ciphers annexed, by the tabular difference taken from column D, the quotient will give seconds, which seconds are to be added to the degrees and minutes already found, in the case of a sine or tangent, but to be subtracted in the case of a cosine or cotangent.

EXAMPLES.

1. Find the arc whose logarithmic sine is 9.365365.

$$9.365365 = \text{given logarithmic sine.}$$

$$\sin. 13^\circ 24' = 9.365016$$

$$349 = \text{difference,}$$

$$\text{and } 34900 \div 883 = 39, \text{ the number of seconds.}$$

Hence, the arc $13^\circ 24' 39''$ corresponds with the logarithmic sine of 9.365365.

2. Find the arc whose cosine is 9.142857.

$$9.142857 = \text{given logarithmic cosine.}$$

$$\cos. 82^\circ 1' = 9.142655$$

$$20200 \div 1500 = 13''.$$

$$\text{subtract } 13''$$

$$82^\circ 0' 47'' = \text{arc required.}$$

3. Find the arc whose tangent is 10.528017.

$$10.528017 = \text{given logarithmic tangent.}$$

$$\text{tang. } 73^\circ 29' = 10.527931$$

$$8600 \div 772 = 11''.$$

$$\text{add } 11''$$

$$73^\circ 29' 11'' = \text{arc required.}$$

4. Find the arc whose cotangent is 9.555555.

$$9.555555 = \text{given logarithmic cotangent.}$$

$$\cotan. 70^\circ 14' = 9.555536$$

$$1900 \div 661 = 3'', \text{ nearly.}$$

$$\text{subtract } 3''$$

$$70^\circ 13' 57'' = \text{arc required.}$$

§ 38. The secants and cosecants are omitted in this table,

since they are so readily found by the aid of the cosines and sines. By equation B, § 8, we have

$$\sec. A = \frac{1}{\cos. A}; \operatorname{cosec.} A = \frac{1}{\sin. A}.$$

Clearing of fractions, we have

$$\sec. A \times \cos. A = 1; \operatorname{cosec.} A \times \sin. A = 1.$$

Taking the logarithms, and observing to add 10 to each logarithm, we have

$$\log. \sec. A + \log. \cos. A = 20; \log. \operatorname{cosec.} A + \log. \sin. A = 20.$$

$$\begin{aligned} \text{Hence,} \quad \log. \sec. A &= 20 - \log. \cos. A, \\ \log. \operatorname{cosec.} A &= 20 - \log. \sin. A. \end{aligned}$$

From this we see that

The logarithmic secant is found by subtracting the logarithmic cosine from 20; and the logarithmic cosecant is found by subtracting the logarithmic sine from 20.

EXAMPLES.

1. Find the logarithmic secant and cosecant of $41^\circ 41'$.

$$\begin{aligned} \sin. \quad 41^\circ 41' &= \overset{20\cdot}{9\cdot822830}; \cos. 41^\circ 41' = \overset{20\cdot}{9\cdot873223}. \\ \operatorname{cosec.} 41^\circ 41' &= 10\cdot177170; \sec. 41^\circ 41' = 10\cdot126777. \end{aligned}$$

2. What is the secant and cosecant of $15^\circ 15' 15''$?

$$\begin{aligned} \sin. \quad 15^\circ 15' 15'' &= \overset{20\cdot}{9\cdot420123}; \cos. 15^\circ 15' 15'' = \overset{20\cdot}{9\cdot984423}. \\ \operatorname{cosec.} 15^\circ 15' 15'' &= 10\cdot579877; \sec. 15^\circ 15' 15'' = 10\cdot015577. \end{aligned}$$

§ 39. Since we have (equation B, § 8) $\operatorname{tang.} A = \frac{1}{\cotan. A}$, it follows that

$$\operatorname{tang.} A \times \cotan. A = 1,$$

$$\text{and} \quad \log. \operatorname{tang.} A + \log. \cotan. A = 20.$$

Hence, the logarithmic cotangent may be found by subtracting the logarithmic tangent from 20; and, conversely, the logarithmic tangent may be found by subtracting the logarithmic cotangent from 20.

EXAMPLES.

1. The logarithmic tangent of an arc is $9\cdot545454$; what is the logarithmic cotangent of the same arc? *Ans.* $10\cdot454546$.

2. The logarithmic cotangent is $9\cdot333444$; what is that of the tangent of the same angle? *Ans.* $10\cdot666556$.

§ 40. In any plane triangle there are three sides and three angles, making in all six parts to be considered. Any three of these six parts, provided one at least is a side, being given or known, the other three parts can be found. Thus, in Book I., Geometry, we have seen that when two triangles had three parts of the one respectively equal to the three corresponding parts of the other, provided the three parts compared were not all angles, the two triangles were identical, or equal in all respects.

The case in which three angles of a triangle are respectively equal to three angles of a second triangle, does not lead of necessity to an equality of these two triangles, but simply to their *similarity*. Hence, when in a triangle three parts, provided one at least is a side, are given, the remaining parts can be found.

§ 41. In the case of right triangles, one angle, that is, the right angle, is always given; consequently two parts in addition to the right angle must be given, one of which must be a side, in order that we may find the remaining parts.

The geometrical or graphic method of solving a triangle when a sufficient number of parts are known, has already been exhibited in the Problems at the end of Book Second of Geometry. The method of calculating the numerical values of these parts—which we now proceed to explain—belongs to Trigonometry.

CHAPTER III.

SOLUTION OF RIGHT TRIANGLES.

CASE I.

§ 42. *Given, the hypotenuse and one of the angles, to find the other parts.*

Suppose the given angle to be A. Then will the angle C be found by subtracting A from 90° .

By our definitions (§ 8), we have

$$\sin. A = \frac{p}{h}; \cos. A = \frac{b}{h},$$

which, cleared of fractions, give

$$p = h \sin. A. \quad . \quad . \quad . \quad (1.)$$

$$b = h \cos. A. \quad . \quad . \quad . \quad (2.)$$

By using logarithms, we have

$$\log. p = \log. h + \log. \sin. A. \quad . \quad (3.)$$

$$\log. b = \log. h + \log. \cos. A. \quad . \quad (4.)$$

As an example, suppose the hypotenuse to be 125, and the angle at A to be $37^\circ 30'$. Required the other parts.

$$h = 125; A = 37^\circ 30'.$$

We find $C = 90^\circ - 37^\circ 30' = 52^\circ 30'$.

BY NATURAL NUMBERS. (TABLE III.)

$$\sin. 37^\circ 30' = 0.60876; \cos. 37^\circ 30' = 0.79335$$

125	125
304380	396675
121752	158670
60876	79335
76.09500 = p.	99.16875 = b.

BY LOGARITHMS.

$$\begin{array}{rcl}
 \log. 125 & = & 2.096910 \\
 \log. \sin. 37^\circ 30' & = & 9.784447 ; \log. \cos. 37^\circ 30' = 9.899467 \\
 \log. p & = & 1.881357 \\
 \log. b & = & 1.996377
 \end{array}$$

Hence, $p = 76.095$, and $b = 99.169$.

NOTE.—In adding the $\log. \sin.$ and $\log. \cos.$ respectively to the $\log. 125$, we rejected 10 from the sum to correct for the 10 which had been added to all logarithms of the sines, cosines, &c. (§ 34).

CASE II.

§ 43. *Given, a side, that is, either the perpendicular or the base, and the angle opposite that side, to find the other parts.*

Suppose the given side to be p , and the given angle A . Then will the angle C be the complement of A .

By definitions (§ 8), we have

$$\sin. A = \frac{p}{h}; \cot. A = \frac{b}{p},$$

which readily give

$$h = \frac{p}{\sin. A}. \quad (1.)$$

$$b = p \cot. A. \quad (2.)$$

By using logarithms, we have

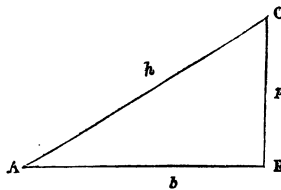
$$\log. h = \log. p - \log. \sin. A. \quad (3.)$$

$$\log. b = \log. p + \log. \cotan. A. \quad (4.)$$

For example, suppose the perpendicular to be 75, and the angle at A to be $39^\circ 15'$. Required the other parts.

$$p = 75; A = 39^\circ 15'.$$

We find $C = 90^\circ - 39^\circ 15' = 50^\circ 45'$.



By NATURAL NUMBERS. (TABLE III.)

$$\sin. 39^\circ 15' = 0.63271; \cotan. 39^\circ 15' = 1.22394$$

$$0.63271) 75.00000 (118.53 = h.$$

63271

117290

63271

540190

506168

340220

316355

238650

189813

48837

1.22394

75

611970

856758

91.79550 = b.

By LOGARITHMS.

$$\log. 75 = 1.875061 \quad . \quad . \quad . \quad 1.875061$$

$$\log. \sin. 39^\circ 15' = 9.801201; \log. \cotan. 39^\circ 15' = 10.087760$$

$$\log. h = 2.073860$$

$$\log. b = 1.962821$$

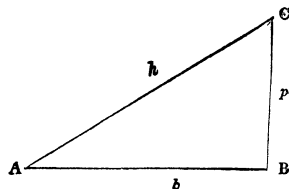
Hence, $h = 118.54$, nearly, and $b = 91.795$, nearly.

NOTE.—In this example, before subtracting the log. of the sine of $39^\circ 15'$ from the log. of 75, it was necessary to increase this logarithm by 10, since the logarithm of the sine had been increased by the same quantity. In the second operation, after adding the log. of the cotangent to the log. of 75, we rejected 10 from the sum, for the reason assigned in the last note.

CASE III.

§ 44. *Given, a side, and the angle adjacent that side, to find the other parts.*

Suppose the given side to be b , and the given angle A . Then will the angle C be the complement of A .



By definitions (§ 8), we have

$$\cos. A = \frac{b}{h}; \text{ tang. } A = \frac{p}{b},$$

which give

$$h = \frac{b}{\cos. A}. \quad . \quad . \quad . \quad . \quad (1.)$$

$$p = b \text{ tang. } A. \quad . \quad . \quad . \quad . \quad (2.)$$

Using logarithms, we have

$$\log. h = \log. b - \log. \cos. A. \quad . \quad . \quad (3.)$$

$$\log. p = \log. b + \log. \text{tang. } A. \quad . \quad . \quad (4.)$$

For example, suppose the base to be 90.5 , and the angle at A to be $50^\circ 30'$. Required the other parts.

$$b = 90.5; A = 50^\circ 30'.$$

We find $C = 90^\circ - 50^\circ 30' = 39^\circ 30'$.

BY NATURAL NUMBERS. (TABLE III.)

$$\cos. 50^\circ 30' = 0.63608; \text{ tang. } 50^\circ 30' = 1.21310.$$

$$0.63608) 90.50000 (142.27 = h$$

63608

268920

254432

144880

127216

176640

127216

494240

445256

48984

1.2131

90.5

60655

109179

109.78555 = p .

BY LOGARITHMS.

$$\log. 90.5 = 1.956649 \quad . \quad . \quad . \quad . \quad 1.956649$$

$$\log. \cos. 50^\circ 30' = 9.803511; \log. \text{tang. } 50^\circ 30' = 10.083896$$

$$\log. h = 2.153138$$

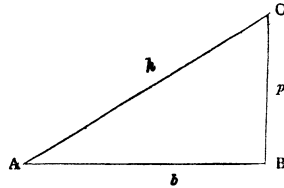
$$\log. p = 2.040545$$

Hence, $h = 142.27$, and $p = 109.78$.

CASE IV.

§ 45. *Given, the hypotenuse and one of the sides, to find the other parts.*

Suppose the given side to be b , then (§ 8) we have



$$\sin. C = \cos. A = \frac{b}{h}, \quad . \quad . \quad . \quad (1.)$$

or, using logarithms,

$$\log. \sin. C = \log. \cos. A = \log. b - \log. h. \quad . \quad (2.)$$

Again, from the fundamental property of the right triangle, we have

$$b^2 + p^2 = h^2,$$

hence

$$\begin{aligned} p^2 &= h^2 - b^2 = (h + b)(h - b), \\ p &= \sqrt{(h + b)(h - b)}. \quad . \quad . \quad . \quad (3.) \end{aligned}$$

Using logarithms, we have

$$\log. p = \frac{1}{2} [\log. (h + b) + \log. (h - b)]. \quad . \quad (4.)$$

For example, suppose the hypotenuse to be 112, and the base to be 97. Required the other parts.

$$h = 112; b = 97.$$

$$112) 97.00000 \quad (0.86607 = \sin. C = \cos. A.$$

$$\begin{array}{r} 896 \\ \hline 740 \\ 672 \\ \hline 680 \\ 672 \\ \hline 800 \\ 784 \\ \hline 16 \end{array}$$

Hence (by Table III.), $C = 60^\circ$ and $A = 30^\circ$.

Again, $h + b = 209$, and $h - b = 15$.

Extracting Square Root.

209	5	3135	(55·99 = p .
15	105	25	
	1109		
1045	11189	635	
209		525	
3135		11000	
		9981	
		101900	
		100701	
		199	

By LOGARITHMS.

$$\log. 97 = 1.986772$$

$$\log. 112 = 2.049218$$

$$9.937554 = \log. \sin. C = \log. \cos. A.$$

Hence, $C = 60^\circ 0' 19''$; $A = 29^\circ 59' 41''$.

Again,

$$\log. 209 = 2.320146$$

$$\log. 15 = 1.176091$$

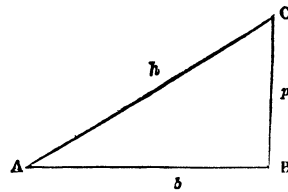
$$2) \underline{3.496237}$$

$$\log. p = 1.748118$$

$$\text{Hence, } p = 55.991.$$

CASE V.

§ 46. *Given, the two sides, to find the other parts.*



By § 8, we have

$$\text{tang. } A = \cotan. C = \frac{p}{b}. \quad . \quad . \quad (1.)$$

or, using logarithms,

$$\log. \text{tang. } A = \log. \cotan. C = \log. p - \log. b. \quad (2.)$$

The hypotenuse is immediately deduced from

$$h^2 = b^2 + p^2,$$

which gives

$$h = \sqrt{b^2 + p^2}. \quad . \quad . \quad . \quad (3.)$$

Since this last formula is not well adapted to the use of logarithms, we will remark that after having found the angles, by means of equations (1) or (2), we may then determine the hypotenuse by Case II., equation (1), which gives

$$\log. h = \log. p - \log. \sin. A. \quad . \quad . \quad . \quad (4.)$$

As an example, suppose the base to be 83.5 and the perpendicular to be 62.25. Required the other parts.

$$b = 83.5; p = 62.25.$$

BY NATURAL NUMBERS. (TABLE III.)

$$83.5) 62.25 (0.74550 = \text{tang. } A = \text{cotan. } C.$$

5845

3800

3340

4600

4175

4250

4175

750

Hence, $A = 36^\circ 42'$

$C = 53^\circ 18'$

$$83.5 = b$$

$$83.5$$

$$4175$$

$$2505$$

$$6680$$

$$6972.25 = b^2$$

$$62.25 = p$$

$$62.25$$

$$31125$$

$$12450$$

$$12450$$

$$37350$$

$$p^2 = 3875.0625$$

$$b^2 = 6972.25$$

$$10847.3125 = b^2 + p^2.$$

$$\sqrt{10847.3125} = 104.15 = h.$$

By LOGARITHMS.

$$\log. 62.25 = 1.794139$$

$$\log. 83.5 = 1.921686$$

$$\log. \text{tang. } A = \log. \text{cotan. } C = 9.872453$$

$$\text{Hence, } A = 36^\circ 42' 18''; C = 53^\circ 17' 42''.$$

Again,

$$\log. 62.25 = 1.794139$$

$$\log. \sin. 36^\circ 42' 18'' = 9.776480$$

$$\log. h = 2.017659$$

$$\text{Hence, } h = 104.15.$$

§ 47. *Additional Examples of Right Triangles.*

1. Given, the hypotenuse, equal 365, and one of the acute angles, equal $33^\circ 12'$, to solve the triangle.

$$\text{Ans. } \begin{cases} \text{The other angle} = 56^\circ 48'. \\ \text{The sides} = \begin{cases} 199.86. \\ 305.42. \end{cases} \end{cases}$$

2. Given, one of the sides equal to 33.33, and the angle opposite this side = $83^\circ 33'$, to solve the triangle.

$$\text{Ans. } \begin{cases} \text{The other angle} = 6^\circ 27'. \\ \text{The other side} = 3.768. \\ \text{The hypotenuse} = 33.542. \end{cases}$$

3. Given, one of the sides equal to 105.5, and the acute angle adjacent this side equal to $46^\circ 3'$, to solve the triangle.

$$\text{Ans. } \begin{cases} \text{The other angle} = 43^\circ 57'. \\ \text{The other side} = 109.44. \\ \text{The hypotenuse} = 152.01. \end{cases}$$

4. Given, the hypotenuse equal 111.1, and one of the sides equal 37.5, to solve the triangle.

$$\text{Ans. } \begin{cases} \text{The other side} = 104.58. \\ \text{The angles} = \begin{cases} 19^\circ 43' 36''. \\ 70^\circ 16' 24''. \end{cases} \end{cases}$$

5. Given, the two sides equal 29.37 and 37.29, to solve the triangle.

$$\text{Ans. } \begin{cases} \text{The hypotenuse} = 47.467. \\ \text{The angles} = \begin{cases} 38^\circ 13' 28''. \\ 51^\circ 46' 32''. \end{cases} \end{cases}$$

The above examples have all been wrought by the use of logarithms, which is, as a general thing, more simple than by

the use of natural numbers only. It will be well, however, for the pupil to apply both methods, as was done in the examples which followed immediately after each case; by so doing, he will be the better prepared to comprehend the true nature of the trigonometrical values as well as their logarithmic values.

§ 48. Hereafter, when we use the word *sine*, *cosine*, *tangent*, or *cotangent*, in connection with logarithmic calculations, we wish to have it understood as meaning the logarithms of those values, increased by 10, as given in Table II. In the usual analytical investigations of trigonometrical formulas, we make use of their real or natural values, as given in Table III.

CHAPTER IV.

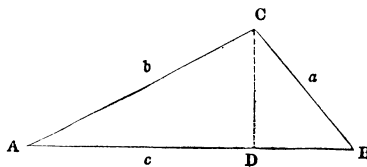
SOLUTION OF OBLIQUE TRIANGLES.

§ 49. ALL the different cases of oblique triangles may be solved by the aid of the following theorems :

THEOREM I.

The sides of any plane triangle are to each other as the sines of their opposite angles. And conversely, the sines of the angles of any plane triangle are to each other as their opposite sides.

In the triangle ABC denote the sides opposite the angles A, B, C, respectively by the letters a, b, c . Draw CD perpendicular to AB, and we shall have (§ 8)



$$\sin. A = \frac{CD}{b}, \sin. B = \frac{CD}{a};$$

dividing the first equation by the second, we have

$$\frac{\sin. A}{\sin. B} = \frac{a}{b}, \quad . \quad . \quad . \quad . \quad (1.)$$

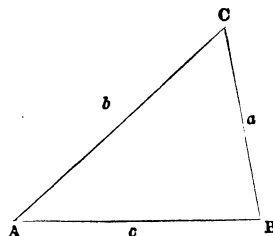
which gives $a : b :: \sin. A : \sin. B. \quad . \quad . \quad . \quad (2.)$

In the same way it may be shown that

$$a : c :: \sin. A : \sin. C. \quad . \quad . \quad . \quad (3.)$$

THEOREM II.

In any plane triangle, the sum of any two sides is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.



By Theorem I., we have

$$b : c :: \sin. B : \sin. C;$$

hence, by composition and division, we have

$$b + c : b - c :: \sin. B + \sin. C : \sin. B - \sin. C,$$

which gives

$$\frac{b+c}{b-c} = \frac{\sin. B + \sin. C}{\sin. B - \sin. C} \quad . \quad . \quad . \quad (1.)$$

By equation (9), § 16, we have

$$\frac{\sin. B + \sin. C}{\sin. B - \sin. C} = \frac{\text{tang. } \frac{1}{2}(B+C)}{\text{tang. } \frac{1}{2}(B-C)}.$$

Hence, we have

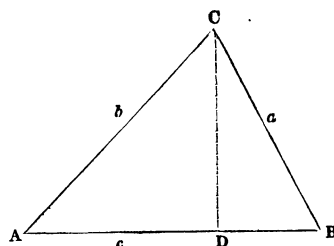
$$\frac{b+c}{b-c} = \frac{\text{tang. } \frac{1}{2}(B+C)}{\text{tang. } \frac{1}{2}(B-C)}, \quad . \quad . \quad . \quad (2.)$$

or,

$$b + c : b - c :: \text{tang. } \frac{1}{2}(B+C) : \text{tang. } \frac{1}{2}(B-C). \quad (3.)$$

THEOREM III.

If from an angle of a plane triangle a perpendicular be drawn so as to meet the opposite side or base, the whole base will be to the sum of the other two sides as the difference of those sides is to the difference of the segments of the base.



Drawing CD perpendicular to AB, we have

$$b^2 = AD^2 + CD^2 \quad . \quad . \quad . \quad (1.)$$

$$a^2 = BD^2 + CD^2. \quad . \quad . \quad . \quad (2.)$$

Subtracting (2) from (1), we have

$$b^2 - a^2 = AD^2 - BD^2,$$

or,

$$(b + a)(b - a) = (AD + BD)(AD - BD), \quad (3.)$$

which, converted into a proportion, will give, observing that $AD + BD = c$,

$$c : b + a :: b - a : AD - BD. \quad (4.)$$

If the triangle has an obtuse angle, then the perpendicular should be drawn from this obtuse angle, otherwise it will not meet the opposite side. The theorem might be so modified as to embrace this case; but as it is used only in the solution of a triangle when the three sides are given, such modification is unnecessary.

§ 50. The solution of all oblique triangles may be included in four cases, as follows :

CASE I.

Given, a side and two angles, to find the other parts.

Since the sum of the three angles of any plane triangle is equal to 180° , the third angle may be found by subtracting the sum of the two given angles from 180° . Having all the angles, the two remaining sides may be found by Theorem I. Thus, if we suppose the side c to be given, we shall have

$$\sin. C : \sin. A :: c : a,$$

$$\sin. C : \sin. B :: c : b,$$

which give

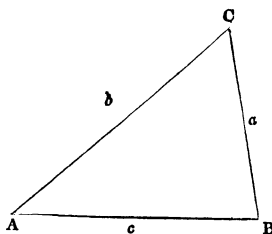
$$a = \frac{c \sin. A}{\sin. C}. \quad (1.)$$

$$b = \frac{c \sin. B}{\sin. C}. \quad (2.)$$

Using logarithms, we have

$$\log. a = \log. c + \log. \sin. A - \log. \sin. C. \quad (3.)$$

$$\log. b = \log. c + \log. \sin. B - \log. \sin. C. \quad (4.)$$



By using arithmetical complements (§ 32), we shall have

$$\log. a = \text{ar. co. log. sin. } C + \log. \sin. A + \log. c \quad (5.)$$

$$\log. b = \text{ar. co. log. sin. } C + \log. \sin. B + \log. c \quad (6.)$$

As an example, suppose the angle $A = 30^\circ 20'$, the angle $B = 50^\circ 10'$, and consequently the angle $C = 99^\circ 30'$. If the side c is 93·37, what will be the lengths of the other sides?

Using equations (5) and (6), we have

$$\begin{array}{rcl} \text{ar. co. sin. } 99^\circ 30' (80^\circ 30') & = & 0\cdot005997 \quad . \quad . \quad 0\cdot005997 \\ \sin. 30^\circ 20' & = & 9\cdot703317; \quad \sin. 50^\circ 10' = 9\cdot885311 \\ \log. 93\cdot37 & = & 1\cdot970207 \quad . \quad . \quad 1\cdot970207 \\ \hline \log. a & = & 1\cdot679521 \quad \log. b = 1\cdot861515 \end{array}$$

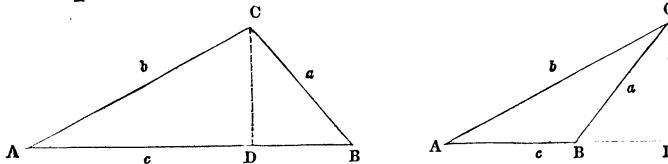
$$\text{Hence, } a = 47\cdot81; \quad b = 72\cdot697.$$

NOTE.—Since the angle $C = 99^\circ 30'$ is an obtuse angle, we subtract it from 180, and obtain its supplement $80^\circ 30'$, the sine of which is 9·994003. Now subtracting this from 10, we have 0·005997 for the arithmetical complement of the logarithmic sine of $99^\circ 30'$.

The arithmetical complement may be readily taken from the table, by beginning at the left hand, and subtracting each figure from 9, except the last significant figure on the right, which must be subtracted from 10. (§ 32.)

CASE II.

Given, two sides and an angle opposite one of them, to find the other parts.



Suppose the sides a and b to be given, and the angle A opposite the side a .

By Theorem I., we have

$$a : b :: \sin. A : \sin. B,$$

$$\text{or,} \quad \sin. B = \frac{b \sin. A}{a}. \quad . \quad . \quad . \quad (1.)$$

Using logarithms, we have

$$\log. \sin. B = \text{ar. co. log. } a + \log. b + \log. \sin. A. \quad (2.)$$

Having thus found the second angle, the third angle at C may be obtained by subtracting the sum of these two from 180° .

Then by Theorem I., we have

$$\sin. A : \sin. C :: a : c,$$

$$\text{or,} \quad c = \frac{a \sin. C}{\sin. A}, \quad . \quad . \quad . \quad . \quad (3.)$$

which by logarithms, becomes

$$\log. c = \text{ar. co. log. sin. A} + \log. \sin. C + \log. a. \quad (4.)$$

SCHOLIUM. In equation (1), the numerator, $b \sin. A$, is the perpendicular CD; now if a is less than this perpendicular, the value of the fraction $\frac{b \sin. A}{a}$, which gives the sine of B, will exceed a unit, which is *impossible*. But while a is greater than this perpendicular, and does not at the same time exceed b , there will be two solutions, as represented in the two diagrams. When there are two solutions, the angle at B in the second triangle will be the supplement of the angle at B in the first.

As an example, suppose the side $a = 75.5$ and the side $b = 98.5$; and suppose the angle at A to be $37^\circ 37'$, required the other parts.

By equation (2), we have

$$\begin{aligned} \text{ar. co. log. } 75.5 &= 8.122053 \\ \log. 98.5 &= 1.993436 \\ \log. \sin. 37^\circ 37' &= 9.785597 \\ \hline \log. \sin. B &= 9.901086 \end{aligned}$$

Hence, $B = 52^\circ 46' 48''$, consequently the angle at C = $89^\circ 36' 12''$.

Now, by equation (4), we have

$$\begin{aligned} \text{ar. co. log. sin. } 37^\circ 37' &= 0.214403 \\ \log. \sin. 89^\circ 36' 12'' &= 9.999989 \\ \log. 75.5 &= 1.877947 \\ \hline \log. c &= 2.092339 \\ \text{and } c &= 123.69. \end{aligned}$$

The above solution corresponds with the first figure; if we take $127^\circ 13' 12''$ for the angle at B, which is the supplement

of $52^\circ 46' 48''$, we shall then find $15^\circ 9' 48''$ for the angle at C. Then, by equation (4), we have

$$\begin{aligned}\text{ar. co. log. sin. } 37^\circ 37' &= 0.214403 \\ \text{log. sin. } 15^\circ 9' 48'' &= 9.417590 \\ \text{log. } 75.5 &= 1.877947 \\ \hline \text{log. } c &= 1.509940 \\ \text{and } c &= 32.355.\end{aligned}$$

This second solution corresponds with the second figure.

CASE III.

Given, two sides and the included angle, to find the other parts.

Suppose the sides b and c , with the included angle A , to be given.

By subtracting the angle A from 180° , which is the sum of the three angles, we have

$$B + C = 180^\circ - A.$$

and

$$\frac{1}{2} (B + C) = 90^\circ - \frac{1}{2} A = \text{complement of } \frac{1}{2} A.$$

Then by Theorem II., we have

$$b + c : b - c :: \cotan. \frac{1}{2} A : \tan. \frac{1}{2} (B - C),$$

which gives

$$\tan. \frac{1}{2} (B - C) = \frac{(b - c) \cotan. \frac{1}{2} A}{b + c}. \quad (1.)$$

Using logarithms, we have

$$\log. \tan. \frac{1}{2} (B - C) = \text{ar. co. log. } (b + c) + \log. (b - c) + \log. \cotan. \frac{1}{2} A. \quad (2.)$$

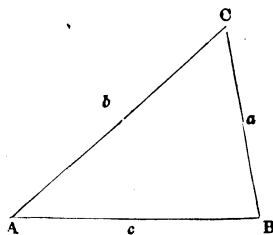
Having thus found half the difference of the angles B and C , we add it to half their sum, and thus obtain the greater angle, which must be opposite the greater side. The smaller angle is found by subtracting half their difference from their half sum.

Again, having found all the angles, we have, by Theorem I.,

$$\sin. B : \sin. A :: b : a,$$

or,

$$a = \frac{b \sin. A}{\sin. B}. \quad (3.)$$



Using logarithms, we have

$$\log. a = \text{ar. co. log. sin. } B + \log. \sin. A + \log. b. \quad (4.)$$

As an example, suppose the side $b = 50.24$, and the side $c = 43.25$, and suppose the angle A to be $40^\circ 15'$, required the other parts.

$$A = 40^\circ 15'; \quad b = 50.24; \quad c = 43.25,$$

consequently,

$$b + c = 93.49; \quad b - c = 6.99; \quad \frac{1}{2} A = 20^\circ 7' 30''.$$

Equation (2) gives

$$\text{ar. co. log. } 93.49 = 8.029235$$

$$\log. 6.99 = 0.844477$$

$$\log. \cotan. 20^\circ 7' 30'' = 10.435993$$

$$\log. \tan. \frac{1}{2} (B - C) = 9.309705$$

$$\text{And } \frac{1}{2} (B - C) = 11^\circ 31' 55''$$

$$\frac{1}{2} (B + C) = 69^\circ 52' 30''$$

$$B = 81^\circ 24' 25'' = \text{sum.}$$

$$C = 58^\circ 20' 35'' = \text{difference.}$$

By equation (4), we have

$$\text{ar. co. log. sin. } 58^\circ 20' 35'' = 0.069965$$

$$\log. \sin. 40^\circ 15' = 9.810316$$

$$\log. 43.25 = 1.635986$$

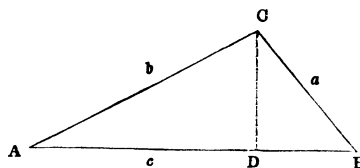
$$\log. a = 1.516267$$

$$\text{And } a = 32.829.$$

CASE IV.

Given, the three sides, to find the angles.

If we conceive CD to be drawn perpendicular to the side c , we shall have, by Theorem III.,



$$c : b + a : : b - a : AD - BD,$$

$$AD - BD = \frac{(b + a)(b - a)}{c}. \quad (1.)$$

Using logarithms, we have

$$\log. (AD - BD) = \text{ar. co. log. } c + \log. (b + a) + \log. (b - a). \quad (2.)$$

This gives half the difference of the segments AD and BD ; if we add this half difference to the half sum, that is, add it to half of c , we shall obtain AD, the greater segment. The half difference subtracted from the half of c will give BD, the shorter segment.

Then solving the two right triangles ADC and BDC, by Case IV. of Right Triangles, we have

$$\cos. A = \frac{AD}{b} ; \cos. B = \frac{BD}{a}.$$

Using logarithms,

$$\log. \cos. A = \log. AD - \log. b, \quad . \quad . \quad (3.)$$

$$\log. \cos. B = \log. BD - \log. a. \quad . \quad . \quad (4.)$$

Having found the angles A and B, subtract their sum from 180° for the third angle C.

For example, suppose we have $a = 50.25$; $b = 60.5$, and $c = 68.4$. Required the angles.

$$b + a = 110.75 ; b - a = 10.25.$$

By equation (2), we have

$$\text{ar. co. log. } 68.4 = 8.164944$$

$$\log. 110.75 = 2.044344$$

$$\log. 10.25 = 1.010724$$

$$\log. (AD - BD) = 1.220012$$

$$AD - BD = 16.596$$

$$\frac{1}{2} c + \frac{1}{2} (AD - BD) = 42.498 = AD.$$

$$\frac{1}{2} c - \frac{1}{2} (AD - BD) = 25.902 = BD.$$

By equations (3) and (4), we have

$$\log. 42.498 = 1.628369 ; \log. 25.902 = 1.413333$$

$$\log. 60.5 = 1.781755 ; \log. 50.25 = 1.701136$$

$$\log. \cos. A = 9.846614 ; \log. \cos. B = 9.712197$$

$$\text{Hence, } A = 45^\circ 22' 35'' ; B = 58^\circ 58' 18''.$$

$$\text{Consequently, } C = 75^\circ 39' 7''.$$

Perhaps a better method for this case would be as follows:—

If to equation (1), which is

$$AD - BD = \frac{b^2 - a^2}{c},$$

we add

$$AD + BD = c,$$

we shall have

$$2 AD = \frac{b^2 + c^2 - a^2}{c};$$

hence,

$$\cos. A = \frac{AD}{b} = \frac{b^2 + c^2 - a^2}{2bc}. \quad . \quad . \quad (5.)$$

Adding 1 to each member of (5), we have

$$1 + \cos. A = 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{b^2 + 2bc + c^2 - a^2}{2bc},$$

or,

$$1 + \cos. A = \frac{(b + c + a)(b + c - a)}{2bc}. \quad . \quad (6.)$$

Denoting half the sum of the three sides by s , we have

$$b + c + a = 2s; \quad b + c - a = 2s - 2a = 2(s - a).$$

Hence equation (6) will become

$$1 + \cos. A = \frac{2s(s - a)}{bc}. \quad . \quad . \quad (7.)$$

By equation (8), § 15, we have

$$1 + \cos. A = 2 (\cos. \frac{1}{2} A)^2.$$

Therefore equation (7) will become

$$2 (\cos. \frac{1}{2} A)^2 = \frac{2s(s - a)}{bc},$$

or,

$$(\cos. \frac{1}{2} A)^2 = \frac{s(s - a)}{bc};$$

consequently,

$$\cos. \frac{1}{2} A = \sqrt{\frac{s(s - a)}{bc}}. \quad . \quad . \quad (8.)$$

In a similar manner, we find

$$\cos. \frac{1}{2} B = \sqrt{\frac{s(s - b)}{ac}}, \quad . \quad . \quad (9.)$$

$$\cos. \frac{1}{2} C = \sqrt{\frac{s(s - c)}{ab}}. \quad . \quad . \quad (10.)$$

By using logarithms, equations (8), (9), and (10) become

$$\left. \begin{aligned} \log. \cos. \frac{1}{2} A &= \frac{1}{2} [\text{ar. co. log. } b + \text{ar. co. log. } c + \log. s + \log. (s-a)] \\ \log. \cos. \frac{1}{2} B &= \frac{1}{2} [\text{ar. co. log. } a + \text{ar. co. log. } c + \log. s + \log. (s-b)] \\ \log. \cos. \frac{1}{2} C &= \frac{1}{2} [\text{ar. co. log. } a + \text{ar. co. log. } b + \log. s + \log. (s-c)] \end{aligned} \right\} \text{(A)}$$

Applying equations (A) to the case already given, we have

$$a=50.25; \quad b=60.5; \quad c=68.4;$$

consequently,

$$\frac{1}{2} (a+b+c) = s = 89.575, \text{ and } s-a = 39.325; \quad s-b = 29.075; \\ s-c = 21.175.$$

$$\begin{array}{rcl} \text{ar. co. log. } b & = & 8.218245; \quad \text{ar. co. log. } a = 8.298864 \\ \text{ar. co. log. } c & = & 8.164944 \quad \text{ar. co. log. } c = 8.164944 \\ \log. s & = & 1.952187 \quad \log. s = 1.952187 \\ \log. (s-a) & = & 1.594669 \quad \log. (s-b) = 1.463520 \\ & & \hline & & 2) 19.930045 \qquad \qquad \qquad 2) 19.879515 \end{array}$$

$$\begin{array}{rcl} \log. \cos. \frac{1}{2} A & = & 9.965022 \quad \log. \cos. \frac{1}{2} B = 9.939757 \\ & & \text{ar. co. log. } a = 8.298864 \\ & & \text{ar. co. log. } b = 8.218245 \\ & & \log. s = 1.952187 \\ & & \log. (s-c) = 1.325823 \\ & & \hline & & 2) 19.795119 \\ \log. \cos. \frac{1}{2} C & = & 9.897559 \end{array}$$

Hence $\frac{1}{2} A = 22^\circ 41' 17''$; $\frac{1}{2} B = 29^\circ 29' 9''$; $\frac{1}{2} C = 37^\circ 49' 34''$,
and $A = 45^\circ 22' 34''$; $B = 58^\circ 58' 18''$; $C = 75^\circ 39' 8''$.

When the sides of a triangle are expressed in small numbers, equation (5) may be advantageously employed: this equation, when the letters are properly permuted so as to apply to each angle, gives

$$\left. \begin{aligned} \text{Nat. cos. } A &= \frac{b^2 + c^2 - a^2}{2bc}, \\ \text{Nat. cos. } B &= \frac{a^2 + c^2 - b^2}{2ac}, \\ \text{Nat. cos. } C &= \frac{a^2 + b^2 - c^2}{2ab}, \end{aligned} \right\} \quad . \quad . \quad \text{(B.)}$$

As an example, suppose the sides to be 5, 7, and 9. Required the angles.

$$a = 5; b = 7; c = 9.$$

$$\text{Nat. cos. A} = \frac{49 + 81 - 25}{2 \times 7 \times 9} = \frac{105}{126},$$

$$\text{Nat. cos. B} = \frac{25 + 81 - 49}{2 \times 5 \times 9} = \frac{57}{90},$$

$$\text{Nat. cos. C} = \frac{25 + 49 - 81}{2 \times 5 \times 7} = \frac{-7}{70}.$$

$$\log. 105 = 2.021189 \qquad \log. 57 = 1.755875$$

$$\log. 126 = 2.100371 \qquad \log. 90 = 1.954243$$

$$\cos. A = 9.920818 \qquad \cos. B = 9.801632$$

$$\log. -7 = 0.845098n$$

$$\log. 70 = 1.845098$$

$$\cos. C = 9.000000n$$

Hence, $A = 33^\circ 33' 27''$; $B = 50^\circ 42' 13''$; $C = 95^\circ 44' 21''$.

NOTE.—In taking the logarithm of -7 , we took it as though it had been $+7$, and at the right annexed the letter n to indicate that the number is negative. So in the value of $\cos. C$ we write the n , thus indicating that the natural cosine of this angle is negative. We then seek in the tables for the angle as though its cosine was positive, and find $84^\circ 15' 39''$, which taken from 180° gives the obtuse angle $95^\circ 44' 21''$ for the true value of C .

§ 51. Additional Examples of Oblique Triangles.

1. Given, two angles of a triangle equal $47^\circ 13'$ and $63^\circ 36'$, and the side opposite the first angle equal 27.5 , to solve the triangle.

$$\text{Ans.} \left\{ \begin{array}{l} \text{The other angle} = 69^\circ 11'. \\ \text{The other sides} = \left\{ \begin{array}{l} 33.562. \\ 35.024. \end{array} \right. \end{array} \right.$$

2. Given, two sides of a triangle equal 47.13 and 63.36 , and the angle opposite the first side equal $27^\circ 50'$, to solve the triangle.

$$\text{First Ans.} \left\{ \begin{array}{l} \text{The other side} = 92.719. \\ \text{The other angles} = \left\{ \begin{array}{l} 38^\circ 52' 47''. \\ 113^\circ 17' 13''. \end{array} \right. \end{array} \right.$$

$$\text{Second Ans.} \left\{ \begin{array}{l} \text{The other side} = 19.341. \\ \text{The other angles} = \left\{ \begin{array}{l} 141^\circ 7' 13''. \\ 11^\circ 2' 47''. \end{array} \right. \end{array} \right.$$

3. Given, two sides of a triangle equal $49\cdot5$ and $101\cdot5$, and the angle opposite the first side equal $30^\circ 25'$, to solve the triangle.

Ans. The question is impossible.

4. Given, two sides of a triangle equal 630 and 800, and the angle opposite the first side equal 100° , to solve the triangle.

Ans. The question is impossible.

5. Given, two sides of a triangle, $77\cdot5$ and $90\cdot25$, and the included angle equal $83^\circ 38'$, to solve the triangle.

$$\text{Ans. } \begin{cases} \text{The other side} &= 112\cdot25. \\ \text{The other angles} &= \begin{cases} 43^\circ 19' 38''. \\ 53^\circ 2' 22''. \end{cases} \end{cases}$$

6. Given, the three sides of a triangle, 40, 50, and 60, to find the angles.

$$\text{Ans. The angles} = \begin{cases} 41^\circ 24' 35''. \\ 55^\circ 46' 16''. \\ 82^\circ 49' 9''. \end{cases}$$

7. Given, two sides of a triangle, 100 and 90, and the angle opposite the first side, $111^\circ 15'$, to solve the triangle.

$$\text{Ans. } \begin{cases} \text{The other side} &= 21\cdot823. \\ \text{The other angles} &= \begin{cases} 11^\circ 44' 7''. \\ 57^\circ 0' 53''. \end{cases} \end{cases}$$

8. In an obtuse-angled triangle one of the acute angles is $29^\circ 15'$, the other acute angle is $2^\circ 45'$; the side opposite the first angle is 6·3. What are the other parts?

$$\text{Ans. } \begin{cases} \text{The obtuse angle} &= 148^\circ. \\ \text{The other sides} &= \begin{cases} 0\cdot619. \\ 6\cdot832. \end{cases} \end{cases}$$

9. Given, two sides of a triangle, $332\cdot21$, $237\cdot61$, and their contained angle equal to $72^\circ 29' 48''$, to find the other parts.

$$\text{Ans. } \begin{cases} \text{The other angles} &= \begin{cases} 40^\circ 59' 35''. \\ 66^\circ 30' 37''. \end{cases} \\ \text{The other side} &= 345\cdot46. \end{cases}$$

10. The three sides of a triangle are 10, 15, and 20. What are the angles?

$$\text{Ans. } \begin{cases} 28^\circ 57' 18''. \\ 46^\circ 34' 3''. \\ 104^\circ 28' 39''. \end{cases}$$

11. The three sides of a triangle are $123\cdot48$, $135\cdot61$, $140\cdot91$. What are the angles?

$$\text{Ans. } \begin{cases} 53^\circ 0' 10''. \\ 61^\circ 17' 50''. \\ 65^\circ 42' 0''. \end{cases}$$

12. Two sides of a triangle are 59·34 and 12·13, and their included angle is $150^\circ 38'$. What are the other parts?

$$\text{Ans. } \left\{ \begin{array}{l} \text{Angles} = \left\{ \begin{array}{l} 24^\circ 30' 12'' \\ 4^\circ 51' 48'' \end{array} \right. \\ \text{Side} = 70\cdot165. \end{array} \right.$$

CHAPTER V.

SPHERICAL TRIGONOMETRY.

§ 52. SPHERICAL TRIGONOMETRY treats of the methods of computing the unknown parts of a spherical triangle, when certain parts are given. But before proceeding to the investigation of the relation of the different parts of a spherical triangle, we will give some

GENERAL TRIGONOMETRIC FORMULAS.

§ 53. Under § 15 and § 16 we have already given several general and useful formulas which were needed in Plane Trigonometry. We now propose to continue these formulas to a greater extent, giving such additional ones as are needed in Spherical Trigonometry, as well as others which will be found convenient in the reduction and simplification of trigonometric expressions in general.

Dividing (10) by (11), § 15, we have

$$\frac{\sin. a}{\cos. a} = \tan. a = \frac{\sqrt{1 - \cos. 2a}}{\sqrt{1 + \cos. 2a}}. \quad . \quad . \quad (1.)$$

Multiplying both the numerator and denominator of the fraction constituting the right-hand member of (1) by the numerator, we have

$$\tan. a = \frac{1 - \cos. 2a}{\sqrt{1 - \cos.^2 2a}} = \frac{1 - \cos. 2a}{\sin. 2a}. \quad . \quad . \quad (2.)$$

If we multiply both the numerator and denominator of the right-hand member of (1) by the denominator, we shall obtain

$$\tan. a = \frac{\sqrt{1 - \cos.^2 2a}}{1 + \cos. 2a} = \frac{\sin. 2a}{1 + \cos. 2a}. \quad . \quad . \quad (3.)$$

Taking the reciprocals of (2) and (3), we have

$$\cot. a = \frac{\sin. 2a}{1 - \cos. 2a} = \frac{1 + \cos. 2a}{\sin. 2a}. \quad . \quad . \quad (4.)$$

We have already, § 15, deduced

$$\tan. (a + b) = \frac{\tan. a + \tan. b}{1 - \tan. a \tan. b}. \quad . \quad . \quad (5.)$$

By a similar process, we find

$$\tan. (a - b) = \frac{\tan. a - \tan. b}{1 + \tan. a \tan. b}. \quad . \quad . \quad (6.)$$

Reciprocating, we have

$$\cot. (a + b) = \frac{1 - \tan. a \tan. b}{\tan. a + \tan. b}. \quad . \quad . \quad (7.)$$

$$\cot. (a - b) = \frac{1 + \tan. a \tan. b}{\tan. a - \tan. b}. \quad . \quad . \quad (8.)$$

If we take $b = a$ in (5) and (7), we shall have

$$\tan. 2a = \frac{2 \tan. a}{1 - \tan.^2 a}; \quad . \quad . \quad (9.)$$

$$\cot. 2a = \frac{1 - \tan.^2 a}{2 \tan. a}. \quad . \quad . \quad (10.)$$

If we take $a = \frac{1}{2}(p + q)$, equation (5) of § 15 will become

$$\sin. (p + q) = 2 \sin. \frac{1}{2}(p + q) \cos. \frac{1}{2}(p + q). \quad (11.)$$

Using this in connection with (5), (6), (7), and (8) of § 16, we readily obtain, by division and reduction, as follows:

$$\frac{\sin. p + \sin. q}{\sin. p - \sin. q} = \frac{\tan. \frac{1}{2}(p + q)}{\tan. \frac{1}{2}(p - q)}; \quad . \quad . \quad (12.)$$

$$\frac{\cos. p + \cos. q}{\cos. q - \cos. p} = \frac{\cot. \frac{1}{2}(p + q)}{\tan. \frac{1}{2}(p - q)}; \quad . \quad . \quad (13.)$$

$$\frac{\sin. p + \sin. q}{\sin. (p + q)} = \frac{\cos. \frac{1}{2}(p - q)}{\cos. \frac{1}{2}(p + q)}; \quad . \quad . \quad (14.)$$

$$\frac{\sin. p - \sin. q}{\sin. (p + q)} = \frac{\sin. \frac{1}{2}(p - q)}{\sin. \frac{1}{2}(p + q)}; \quad . \quad . \quad (15.)$$

$$\frac{\sin. p + \sin. q}{\cos. p + \cos. q} = \tan. \frac{1}{2}(p + q); \quad . \quad . \quad (16.)$$

$$\frac{\sin. p + \sin. q}{\cos. q - \cos. p} = \cot. \frac{1}{2}(p - q); \quad . \quad . \quad (17.)$$

$$\frac{\sin. p - \sin. q}{\cos. p + \cos. q} = \tan. \frac{1}{2}(p - q); \quad . \quad . \quad (18.)$$

$$\frac{\sin. p - \sin. q}{\cos. q - \cos. p} = \cot. \frac{1}{2}(p + q). \quad . \quad . \quad (19.)$$

The following are readily obtained, and are of frequent use :

$$\frac{\sin. (p \pm q)}{\sin. p \sin. q} = \cot. q \pm \cot. p; \quad . \quad . \quad (20.)$$

$$\frac{\cos. (p \pm q)}{\sin. p \sin. q} = \cot. p \cot. q \mp 1; \quad . \quad . \quad (21.)$$

$$\frac{\sin. (p \pm q)}{\cos. p \cos. q} = \tan. p \pm \tan. q; \quad . \quad . \quad (22.)$$

$$\frac{\cos. (p \pm q)}{\cos. p \cos. q} = 1 \mp \tan. p \tan. q; \quad . \quad . \quad (23.)$$

$$\frac{\sin. (p \pm q)}{\sin. p \cos. q} = 1 \pm \cot. p \tan. q; \quad . \quad . \quad (24.)$$

$$\frac{\cos. (p \pm q)}{\sin. p \cos. q} = \cot. p \mp \tan. q; \quad . \quad . \quad (25.)$$

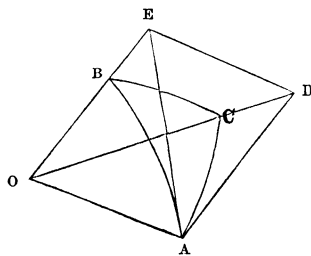
§ 54. We now proceed to investigate the relation of the several parts of a spherical triangle. We shall confine our attention to such triangles only as have been treated of in Book VIII. of Geometry, namely, those whose sides and angles are each less than 180° .

§ 55. *To express the cosine of an angle of a spherical triangle in terms of the sines and cosines of the sides.*

Let ABC be a spherical triangle described upon the surface of a sphere whose centre is at O.

We will denote the angles of this triangle by the large letters A, B, C, and the sides opposite them by the corresponding small letters a, b, c .

Draw AD tangent to the arc AC, and AE tangent to AB.



Then will the spherical angle A be equal to the angle DAE . (Geom., B. VIII., T. V.)

Draw the radius OC and produce it to meet AD in D ; also, draw the radius OB and produce it to meet AE in E . The radius OA being drawn, will be perpendicular to AD , AE .

Now, by Plane Trigonometry, equation (5), Case IV., § 50, the triangle DAE gives

$$DE^2 = AD^2 + AE^2 - 2AD \times AE \times \cos. DAE. \quad (1.)$$

The triangle DOE , in like manner, gives

$$DE^2 = OD^2 + OE^2 - 2OD \times OE \times \cos. DOE. \quad (2.)$$

Equating the right-hand members of (1) and (2), and dividing each term by the square of the radius of the sphere, we have

$$\left. \begin{aligned} \frac{AD^2}{OA^2} + \frac{AE^2}{OA^2} - 2 \times \frac{AD}{OA} \times \frac{AE}{OA} \times \cos. DAE \\ = \frac{OD^2}{OA^2} + \frac{OE^2}{OA^2} - 2 \times \frac{OD}{OA} \times \frac{OE}{OA} \times \cos. DOE. \end{aligned} \right\} \quad (3.)$$

Now, since the triangles OAD , OAE are right-angled at A (Geom., B. II., T. V., C. I.), we have [(A), § 8]

$$\frac{AD}{OA} = \tan. AOD = \tan. b; \quad \frac{AE}{OA} = \tan. AOE = \tan. c;$$

$$\frac{OD}{OA} = \sec. AOD = \sec. b; \quad \frac{OE}{OA} = \sec. AOE = \sec. c.$$

Hence (3) gives, by observing that the angle $DAE = A$, and that the angle $DOE = a$,

$$\left. \begin{aligned} \tan.^2 b + \tan.^2 c - 2 \tan. b \tan. c \cos. A \\ = \sec.^2 b + \sec.^2 c - 2 \sec. b \sec. c \cos. a. \end{aligned} \right\} \quad (4.)$$

Transposing, and observing that $\sec.^2 b - \tan.^2 b$ and $\sec.^2 c - \tan.^2 c$ are each equal to 1, we have

$$2 + 2 \tan. b \tan. c \cos. A = 2 \sec. b \sec. c \cos. a. \quad (5.)$$

Dividing by 2, and substituting for the tangents and secants their values in terms of sines and cosines [(B), § 8], we have

$$1 + \frac{\sin. b}{\cos. b} \times \frac{\sin. c}{\cos. c} \times \cos. A = \frac{1}{\cos. b} \times \frac{1}{\cos. c} \times \cos. a,$$

which, cleared of fractions, becomes

$$\cos. b \cos. c + \sin. b \sin. c \cos. A = \cos. a. \quad (6.)$$

Hence,

$$\left. \begin{aligned} \cos. A &= \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c} \\ \text{By permuting,} \\ \cos. B &= \frac{\cos. b - \cos. c \cos. a}{\sin. c \sin. a}; \\ \cos. C &= \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b} \end{aligned} \right\} \quad . \quad . \quad (A.)$$

We shall not again have occasion to refer to a diagram for the purpose of obtaining any new relation of the respective parts of a spherical triangle, but shall be able to draw from equation (6) all other relations which may be required. Hence, (6) may be regarded as one of the fundamental relations of Spherical Trigonometry.

§ 56. *To express the cosine of a side of a spherical triangle, in terms of the sines and cosines of the angles.*

Let A, B, C, a, b, c , be the angles and sides of a spherical triangle, and A', B', C', a', b', c' the corresponding quantities of its polar triangle.

Then, by (A), § 55, we have

$$\cos. A' = \frac{\cos. a' - \cos. b' \cos. c'}{\sin. b' \sin. c'}. \quad . \quad . \quad (7.)$$

Now we have these relations (Geom., B. VIII., T. VII.):
 $A' = 180^\circ - a$; $B' = 180^\circ - b$; $C' = 180^\circ - c$; $a' = 180^\circ - A$;
 $b' = 180^\circ - B$; $c' = 180^\circ - C$; hence, (7) becomes

$$\cos. (180^\circ - a) = \frac{\cos. (180^\circ - A) - \cos. (180^\circ - B) \cos. (180^\circ - C)}{\sin. (180^\circ - B) \sin. (180^\circ - C)},$$

which gives

$$\left. \begin{aligned} \cos. a &= \frac{\cos. A + \cos. B \cos. C}{\sin. B \sin. C}; \\ \text{and by permuting,} \\ \cos. b &= \frac{\cos. B + \cos. C \cos. A}{\sin. C \sin. A}; \\ \cos. c &= \frac{\cos. C + \cos. A \cos. B}{\sin. A \sin. B} \end{aligned} \right\} \quad . \quad . \quad (B.)$$

By the application of this property of the *Polar Triangle*, any formula of a spherical triangle may be changed into a similar

formula, where the sides will take the place of angles, and the angles will take the place of sides.

This principle of change may be thus stated: *For the sides write the supplements of the opposite angles, and for the angles write the supplements of the opposite sides.*

§ 57. To express the sine of an angle of a spherical triangle, in terms of the sines of the sides of the triangle.

By (A) § 55, we have

$$\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c}.$$

Therefore

$$\begin{aligned} 1 + \cos. A &= \frac{\cos. a - \cos. b \cos. c + \sin. b \sin. c}{\sin. b \sin. c} = \frac{\cos. a - \cos. (b+c)}{\sin. b \sin. c} \\ &= \frac{2 \sin. \frac{1}{2}(a+b+c) \sin. \frac{1}{2}(-a+b+c)}{\sin. b \sin. c}. \quad [\text{See (8), § 16.}] \end{aligned}$$

$$\begin{aligned} \text{If } s &= \frac{1}{2}(a+b+c) \text{ we shall have } s-a = \frac{1}{2}(-a+b+c); \\ (s-b) &= \frac{1}{2}(a-b+c); \quad s-c = \frac{1}{2}(a+b-c). \end{aligned}$$

We therefore have

$$1 + \cos. A = \frac{2 \sin. s \sin. (s-a)}{\sin. b \sin. c}. \quad (1.)$$

By a similar process we find

$$\begin{aligned} 1 - \cos. A &= \frac{\cos. b \cos. c + \sin. b \sin. c - \cos. a}{\sin. b \sin. c} = \frac{\cos. (b-c) - \cos. a}{\sin. b \sin. c} \\ &= \frac{2 \sin. \frac{1}{2}(a-b+c) \sin. \frac{1}{2}(a+b-c)}{\sin. b \sin. c} = \frac{2 \sin. (s-b) \sin. (s-c)}{\sin. b \sin. c}. \quad (2.) \end{aligned}$$

Taking the product of (1) and (2) we have

$$1 - \cos.^2 A = \sin.^2 A = \frac{4 \sin. s \sin. (s-a) \sin. (s-b) \sin. (s-c)}{\sin.^2 b \sin.^2 c},$$

and

$$\left. \begin{aligned} \sin. A &= \frac{2}{\sin. b \sin. c} \left\{ \sin. s \sin. (s-a) \sin. (s-b) \sin. (s-c) \right\}^{\frac{1}{2}}; \\ \text{By permuting,} \\ \sin. B &= \frac{2}{\sin. c \sin. a} \left\{ \sin. s \sin. (s-a) \sin. (s-b) \sin. (s-c) \right\}^{\frac{1}{2}}; \\ \sin. C &= \frac{2}{\sin. a \sin. b} \left\{ \sin. s \sin. (s-a) \sin. (s-b) \sin. (s-c) \right\}^{\frac{1}{2}}. \end{aligned} \right\} (C.)$$

By (1) we have

$$1 + \cos. A = 2 \cos.^2 \frac{1}{2} A = \frac{2 \sin. s \sin. (s-a)}{\sin. b \sin. c};$$

consequently,

$$\left. \begin{aligned} \cos. \frac{1}{2} A &= \left\{ \frac{\sin. s \sin. (s-a)}{\sin. b \sin. c} \right\}^{\frac{1}{2}}; \\ \text{And permuting,} \\ \cos. \frac{1}{2} B &= \left\{ \frac{\sin. s \sin. (s-b)}{\sin. c \sin. a} \right\}^{\frac{1}{2}}; \\ \cos. \frac{1}{2} C &= \left\{ \frac{\sin. s \sin. (s-c)}{\sin. a \sin. b} \right\}^{\frac{1}{2}}. \end{aligned} \right\} \quad (C')$$

By (2) we have

$$1 - \cos. A = 2 \sin.^2 \frac{1}{2} A = \frac{2 \sin. (s-b) \sin. (s-c)}{\sin. b \sin. c}.$$

Consequently,

$$\left. \begin{aligned} \sin. \frac{1}{2} A &= \left\{ \frac{\sin. (s-b) \sin. (s-c)}{\sin. b \sin. c} \right\}^{\frac{1}{2}}; \\ \text{And permuting,} \\ \sin. \frac{1}{2} B &= \left\{ \frac{\sin. (s-c) \sin. (s-a)}{\sin. c \sin. a} \right\}^{\frac{1}{2}}; \\ \sin. \frac{1}{2} C &= \left\{ \frac{\sin. (s-a) \sin. (s-b)}{\sin. a \sin. b} \right\}^{\frac{1}{2}}. \end{aligned} \right\} \quad (C'')$$

Finally, dividing the expressions (C'') by those of (C') we have

$$\left. \begin{aligned} \tan. \frac{1}{2} A &= \left\{ \frac{\sin. (s-b) \sin. (s-c)}{\sin. s \sin. (s-a)} \right\}^{\frac{1}{2}}; \\ \tan. \frac{1}{2} B &= \left\{ \frac{\sin. (s-c) \sin. (s-a)}{\sin. s \sin. (s-b)} \right\}^{\frac{1}{2}}; \\ \tan. \frac{1}{2} C &= \left\{ \frac{\sin. (s-a) \sin. (s-b)}{\sin. s \sin. (s-c)} \right\}^{\frac{1}{2}}. \end{aligned} \right\} \quad (C''')$$

§ 58. *To express the sine of a side of a spherical triangle, in terms of the sines and cosines of the angles.*

By (B) § 56, we have

$$\cos. a = \frac{\cos. A + \cos. B \cos. C}{\sin. B \sin. C};$$

consequently,

$$1 + \cos. a = \frac{\cos. A + \cos. B \cos. C + \sin. B \sin. C}{\sin. B \sin. C} = \frac{\cos. A + \cos. (B-C)}{\sin. B \sin. C} \\ = \frac{2 \cos. \frac{1}{2}(A+B-C) \cos. \frac{1}{2}(A-B+C)}{\sin. B \sin. C}.$$

If $S = \frac{1}{2}(A+B+C)$ we shall have $S-A = \frac{1}{2}(-A+B+C)$; $S-B = \frac{1}{2}(A-B+C)$; $S-C = \frac{1}{2}(A+B-C)$. We therefore have

$$1 + \cos. a = \frac{2 \cos. (S-B) \cos. (S-C)}{\sin. B \sin. C}. \quad (1.)$$

By a similar process we find

$$1 - \cos. a = -\frac{\cos. B \cos. C - \sin. B \sin. C + \cos. A}{\sin. B \sin. C} \\ = -\frac{\cos. (B+C) + \cos. A}{\sin. B \sin. C} = -\frac{2 \cos. \frac{1}{2}(A+B+C) \cos. \frac{1}{2}(-A+B+C)}{\sin. B \sin. C} \\ = -\frac{2 \cos. S \cos. (S-A)}{\sin. B \sin. C}. \quad (2.)$$

Taking the product of (1) and (2), we have

$$1 - \cos.^2 a = \sin.^2 a = -\frac{4 \cos. S \cos. (S-A) \cos. (S-B) \cos. (S-C)}{\sin.^2 B \sin.^2 C}.$$

And

$$\sin. a = \frac{2}{\sin. B \sin. C} \left\{ -\cos. S \cos. (S-A) \cos. (S-B) \cos. (S-C) \right\}^{\frac{1}{2}}; \\ \text{Permuting,} \\ \sin. b = \frac{2}{\sin. C \sin. A} \left\{ -\cos. S \cos. (S-A) \cos. (S-B) \cos. (S-C) \right\}^{\frac{1}{2}}; \\ \sin. c = \frac{2}{\sin. A \sin. B} \left\{ -\cos. S \cos. (S-A) \cos. (S-B) \cos. (S-C) \right\}^{\frac{1}{2}}. \quad (D.)$$

By (1) we have

$$1 + \cos. a = 2 \cos.^2 \frac{1}{2} a = \frac{2 \cos. (S-B) \cos. (S-C)}{\sin. B \sin. C}.$$

Consequently,

$$\cos. \frac{1}{2} a = \left\{ \frac{\cos. (S-B) \cos. (S-C)}{\sin. B \sin. C} \right\}^{\frac{1}{2}}; \\ \cos. \frac{1}{2} b = \left\{ \frac{\cos. (S-C) \cos. (S-A)}{\sin. C \sin. A} \right\}^{\frac{1}{2}}; \\ \cos. \frac{1}{2} c = \left\{ \frac{\cos. (S-A) \cos. (S-B)}{\sin. A \sin. B} \right\}^{\frac{1}{2}}. \quad (D')$$

By (2) we have

$$1 - \cos. a = 2 \sin.^2 \frac{1}{2} a = -\frac{2 \cos. S \cos. (S-A)}{\sin. B \sin. C}.$$

Consequently,

$$\left. \begin{aligned} \sin. \frac{1}{2} a &= \left\{ \frac{-\cos. S \cos. (S-A)}{\sin. B \sin. C} \right\}^{\frac{1}{2}}; \\ \sin. \frac{1}{2} b &= \left\{ \frac{-\cos. S \cos. (S-B)}{\sin. C \sin. A} \right\}^{\frac{1}{2}}; \\ \sin. \frac{1}{2} c &= \left\{ \frac{-\cos. S \cos. (S-C)}{\sin. A \sin. B} \right\}^{\frac{1}{2}}. \end{aligned} \right\} \quad (D'')$$

Finally, dividing the expressions (D'') by those of (D'), we have

$$\left. \begin{aligned} \tan. \frac{1}{2} a &= \left\{ \frac{-\cos. S \cos. (S-A)}{\cos. (S-B) \cos. (S-C)} \right\}^{\frac{1}{2}}; \\ \tan. \frac{1}{2} b &= \left\{ \frac{-\cos. S \cos. (S-B)}{\cos. (S-C) \cos. (S-A)} \right\}^{\frac{1}{2}}; \\ \tan. \frac{1}{2} c &= \left\{ \frac{-\cos. S \cos. (S-C)}{\cos. (S-A) \cos. (S-B)} \right\}^{\frac{1}{2}}. \end{aligned} \right\} \quad (D''')$$

Groups marked (D), (D'), (D''), and (D''') might have been immediately deduced from those marked respectively (C), (C'), (C''), and (C'''), by the application of the principle of the *Polar Triangle*, as indicated under § 56.

Since the negative sign precedes $\cos. S$ in the expressions under the radicals of (D), (D''), and (D''') it might, at first view, be supposed that these values were under an impossible form. It is however easily shown that this is not the case. For (Geom. B. VIII., T. XV.) we know that the sum of all the angles of a spherical triangle is greater than two right angles, and less than six right angles.

Consequently,

$$A + B + C > 180^\circ \text{ and } < 540^\circ.$$

And

$$\frac{1}{2} (A + B + C) \text{ or } S > 90^\circ \text{ and } < 270^\circ.$$

Hence, the cosine of S is always negative, and $-\cos. S$ is therefore always positive.

Again, if a', b', c' be the three sides of the polar triangle, since the sum of any two sides of a spherical triangle is greater

than the third side (Geom. B. VIII., T. I), we have $b' + c' > a'$; that is,

$$180^\circ - B + 180^\circ - C > 180^\circ - A;$$

consequently,

$$-A + B + C < 180^\circ,$$

and

$$\frac{1}{2}(-A + B + C) < 90^\circ;$$

hence, $\cos. (S-A)$ is always positive, and in like manner $\cos. (S-B)$, $\cos. (S-C)$, are always positive; hence the expressions (D), (D'), (D'') are in every case possible.

§ 59. *The sines of the angles of a spherical triangle are to each other as the sines of their opposite sides.*

Expressions (C) immediately give by division,

$$\left. \begin{aligned} \frac{\sin. A}{\sin. B} &= \frac{\sin. c \sin. a}{\sin. b \sin. c} = \frac{\sin. a}{\sin. b}; \\ \frac{\sin. A}{\sin. C} &= \frac{\sin. a \sin. b}{\sin. b \sin. c} = \frac{\sin. a}{\sin. c}; \\ \frac{\sin. B}{\sin. C} &= \frac{\sin. a \sin. b}{\sin. c \sin. a} = \frac{\sin. b}{\sin. c}. \end{aligned} \right\} \quad . \quad . \quad (E.)$$

§ 60. *To express the tangent of the sum and difference of two angles of a spherical triangle, in terms of the sides opposite to these angles, and the third angle of the triangle.*

By (A) § 55, we have from the third expression,

$$\cos. c = \cos. a \cos. b + \sin. a \sin. b \cos. C. \quad . \quad (1.)$$

Substituting this value of $\cos. c$ in the first expression of (A), we find

$$\begin{aligned} \cos. A &= \frac{\cos. a - \cos. a \cos.^2 b - \cos. b \sin. a \sin. b \cos. C}{\sin. b \sin. c} \\ &= \frac{\cos. a (1 - \cos.^2 b) - \cos. b \sin. a \sin. b \cos. C}{\sin. b \sin. c} \\ &= \frac{\cos. a \sin. b - \cos. b \sin. a \cos. C}{\sin. c}. \quad . \quad (2.) \end{aligned}$$

If we substitute the value of $\cos. c$, given by (1), in the second expression of (A), we shall obtain

$$\cos. B = \frac{\cos. b \sin. a - \cos. a \sin. b \cos. C}{\sin. c}. \quad . \quad (3.)$$

Adding (2) and (3), we have

$$\begin{aligned}\cos. A + \cos. B &= \frac{\sin. a \cos. b + \cos. a \sin. b - (\sin. a \cos. b + \cos. a \sin. b) \cos. C}{\sin. c} \\ &= \frac{\sin. (a + b) - \sin. (a + b) \cos. C}{\sin. c} = \frac{\sin. (a + b) (1 - \cos. C)}{\sin. c}. \quad (4.)\end{aligned}$$

Again, we have, by the first equation of (E),

$$\frac{\sin. A}{\sin. B} = \frac{\sin. a}{\sin. b};$$

consequently,

$$\frac{\sin. A \pm \sin. B}{\sin. B} = \frac{\sin. a \pm \sin. b}{\sin. b}.$$

Hence,

$$\sin. A \pm \sin. B = (\sin. a \pm \sin. b) \frac{\sin. B}{\sin. b} = (\sin. a \pm \sin. b) \frac{\sin. C}{\sin. c}. \quad (5.)$$

Dividing (5) by (4), using first the positive sign, we have

$$\frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \frac{\sin. a + \sin. b}{\sin. (a + b)} \times \frac{\sin. C}{1 - \cos. C};$$

hence, by equations (16), (14), and (4) of § 53, this becomes

$$\tan. \frac{1}{2} (A + B) = \frac{\cos. \frac{1}{2} (a - b)}{\cos. \frac{1}{2} (a + b)} \times \cot. \frac{1}{2} C. \quad (6.)$$

In a similar manner, by dividing (5), with the negative sign, by (4), we find

$$\frac{\sin. A - \sin. B}{\cos. A + \cos. B} = \frac{\sin. a - \sin. b}{\sin. (a + b)} \times \frac{\sin. C}{1 - \cos. C},$$

which, by equations (18), (15), and (4) of § 53, becomes

$$\tan. \frac{1}{2} (A - B) = \frac{\sin. \frac{1}{2} (a - b)}{\sin. \frac{1}{2} (a + b)} \times \cot. \frac{1}{2} C. \quad (7.)$$

Equations (6) and (7) and their permuted values give as follows :

$$\left. \begin{aligned} \left\{ \begin{aligned} \tan. \frac{1}{2}(A+B) &= \frac{\cos. \frac{1}{2}(a-b)}{\cos. \frac{1}{2}(a+b)} \times \cot. \frac{1}{2} C; \\ \tan. \frac{1}{2}(A-B) &= \frac{\sin. \frac{1}{2}(a-b)}{\sin. \frac{1}{2}(a+b)} \times \cot. \frac{1}{2} C. \end{aligned} \right. \\ \left\{ \begin{aligned} \tan. \frac{1}{2}(B+C) &= \frac{\cos. \frac{1}{2}(b-c)}{\cos. \frac{1}{2}(b+c)} \times \cot. \frac{1}{2} A; \\ \tan. \frac{1}{2}(B-C) &= \frac{\sin. \frac{1}{2}(b-c)}{\sin. \frac{1}{2}(b+c)} \times \cot. \frac{1}{2} A. \end{aligned} \right. \\ \left\{ \begin{aligned} \tan. \frac{1}{2}(C+A) &= \frac{\cos. \frac{1}{2}(c-a)}{\cos. \frac{1}{2}(c+a)} \times \cot. \frac{1}{2} B; \\ \tan. \frac{1}{2}(C-A) &= \frac{\sin. \frac{1}{2}(c-a)}{\sin. \frac{1}{2}(c+a)} \times \cot. \frac{1}{2} B. \end{aligned} \right. \end{aligned} \right\} \quad (F.)$$

If we convert equations (1) and (2) of (F) into proportions, we have

$$\begin{aligned} \cos. \frac{1}{2}(a+b) : \cos. \frac{1}{2}(a-b) :: \cot. \frac{1}{2} C : \tan. \frac{1}{2}(A+B), \\ \sin. \frac{1}{2}(a+b) : \sin. \frac{1}{2}(a-b) :: \cot. \frac{1}{2} C : \tan. \frac{1}{2}(A-B); \end{aligned}$$

which proportions are known as *Napier's first and second Analogies*. They are used in solving a triangle when two sides and the included angle are given.

§ 61. *To express the tangent of the sum and difference of two sides of a spherical triangle, in terms of the angles opposite to them, and the third side of the triangle.*

Let A, B, C, a, b, c be the angles and sides of a spherical triangle, A', B', C', a', b', c' , the corresponding parts of the polar triangle; then by (F), we have

$$\tan. \frac{1}{2}(A' + B') = \frac{\cos. \frac{1}{2}(a' - b')}{\cos. \frac{1}{2}(a' + b')} \times \cot. \frac{1}{2} C'.$$

Therefore,

$$\tan. \frac{1}{2} [(180^\circ - a) + (180^\circ - b)] = \frac{\cos. \frac{1}{2} [(180^\circ - A) - (180^\circ - B)]}{\cos. \frac{1}{2} [(180^\circ - A) + (180^\circ - B)]} \times \cot. \frac{1}{2} (180^\circ - c).$$

This becomes

$$\tan. \frac{1}{2}(a+b) = \frac{\cos. \frac{1}{2}(A-B)}{\cos. \frac{1}{2}(A+B)} \times \tan. \frac{1}{2} c. \quad (1.)$$

Similarly we have

$$\tan. \frac{1}{2}(A' - B') = \frac{\sin. \frac{1}{2}(a' - b')}{\sin. \frac{1}{2}(a' + b')} \times \cot. \frac{1}{2} C'.$$

Therefore,

$$\tan. \frac{1}{2} [(180^\circ - a) - (180^\circ - b)] = \frac{\sin. \frac{1}{2} [(180^\circ - A) - (180^\circ - B)]}{\sin. \frac{1}{2} [(180^\circ - A) + (180^\circ - B)]} \times \cot. \frac{1}{2} (180^\circ - c),$$

which becomes

$$\tan. \frac{1}{2}(a - b) = \frac{\sin. \frac{1}{2}(A - B)}{\sin. \frac{1}{2}(A + B)} \times \tan. \frac{1}{2} c. \quad (2.)$$

Equations (1) and (2), together with their permuted values, give as follows:

$$\left\{ \begin{array}{l} \tan. \frac{1}{2}(a + b) = \frac{\cos. \frac{1}{2}(A - B)}{\cos. \frac{1}{2}(A + B)} \times \tan. \frac{1}{2} c; \\ \tan. \frac{1}{2}(a - b) = \frac{\sin. \frac{1}{2}(A - B)}{\sin. \frac{1}{2}(A + B)} \times \tan. \frac{1}{2} c. \end{array} \right\} \quad (G.)$$

$$\left\{ \begin{array}{l} \tan. \frac{1}{2}(b + c) = \frac{\cos. \frac{1}{2}(B - C)}{\cos. \frac{1}{2}(B + C)} \times \tan. \frac{1}{2} a; \\ \tan. \frac{1}{2}(b - c) = \frac{\sin. \frac{1}{2}(B - C)}{\sin. \frac{1}{2}(B + C)} \times \tan. \frac{1}{2} a. \end{array} \right\}$$

$$\left\{ \begin{array}{l} \tan. \frac{1}{2}(c + a) = \frac{\cos. \frac{1}{2}(C - A)}{\cos. \frac{1}{2}(C + A)} \times \tan. \frac{1}{2} b; \\ \tan. \frac{1}{2}(c - a) = \frac{\sin. \frac{1}{2}(C - A)}{\sin. \frac{1}{2}(C + A)} \times \tan. \frac{1}{2} b. \end{array} \right\}$$

The first and second conditions of (G) being converted into proportions, gives *Napier's third and fourth Analogies*, as follows:

$$\begin{aligned} \cos. \frac{1}{2}(A + B) : \cos. \frac{1}{2}(A - B) &:: \tan. \frac{1}{2} c : \tan. \frac{1}{2}(a + b), \\ \sin. \frac{1}{2}(A + B) : \sin. \frac{1}{2}(A - B) &:: \tan. \frac{1}{2} c : \tan. \frac{1}{2}(a - b). \end{aligned}$$

These analogies are employed in the solution of a triangle when two angles and the interjacent side are given.

§ 62. To express the cotangent of an angle of a spherical triangle, in terms of the side opposite, one of the other sides, and the angle included between these two sides.

Equation (2) of § 60, when cleared of fractions, becomes

$$\cos. A \sin. c = \cos. a \sin. b - \cos. b \sin. a \cos. C. \quad (1.)$$

By (E) we have

$$\sin. A \sin. c = \sin. C \sin. a. \quad . \quad . \quad (2.)$$

Dividing (1) by (2), we find

$$\cot. A = \cot. a \sin. b \operatorname{cosec}. C - \cos. b \cot. C. \quad . \quad (3.)$$

By interchanging B and C, b and c , (3) will give

$$\cot. A = \cot. a \sin. c \operatorname{cosec}. B - \cos. c \cot. B. \quad . \quad (4.)$$

Proceeding in like manner for the other angles, we shall obtain the following group :

$$\left. \begin{aligned} \cot. A &= \cot. a \sin. b \operatorname{cosec}. C - \cos. b \cot. C \\ &= \cot. a \sin. c \operatorname{cosec}. B - \cos. c \cot. B; \\ \cot. B &= \cot. b \sin. c \operatorname{cosec}. A - \cos. c \cot. A \\ &= \cot. b \sin. a \operatorname{cosec}. C - \cos. a \cot. C; \\ \cot. C &= \cot. c \sin. a \operatorname{cosec}. B - \cos. a \cot. B \\ &= \cot. c \sin. b \operatorname{cosec}. A - \cos. b \cot. A. \end{aligned} \right\} \quad (H.)$$

These six equations are of such a nature that they do not yield any new relations by the application of the *principle of the Polar Triangle* (§ 56).

§ 63. By aid of the eight groups of formulas designated by (A), (B), (C), (D), (E), (F), (G), (H), we shall be enabled to solve all the cases of spherical triangles, whether right-angled or oblique-angled. We shall, in the next chapter, proceed to apply these formulas.

§ 64. Before, however, passing to the solution of spherical triangles, we will deduce some other general formulas of *spherics*, which will frequently be found useful in Astronomy.

Equation (6) of § 55, becomes, by permuting,

$$\cos. c \cos. a + \sin. c \sin. a \cos. B = \cos. b. \quad . \quad . \quad (1.)$$

Multiplying (6) of § 55, by $\cos. c$, it becomes

$$\cos. c \cos. a = \cos. b \cos.^2 c + \sin. b \sin. c \cos. c \cos. A. \quad (2.)$$

Subtracting (2) from (1) and substituting $\sin.^2 c$ for $1 - \cos.^2 c$, we have

$$\sin. c \sin. a \cos. B = \cos. b \sin.^2 c - \sin. b \sin. c \cos. c \cos. A;$$

which, divided by $\sin. c$, becomes

$$\sin. a \cos. B = \cos. b \sin. c - \sin. b \cos. c \cos. A. \quad (3.)$$

Interchanging B, C, and b, c , (3) becomes

$$\sin. a \cos. C = \cos. c \sin. b - \sin. c \cos. b \cos. A. \quad (4.)$$

Equations (3) and (4) with their permutations, give

$$\left. \begin{aligned} \sin. a \cos. B &= \cos. b \sin. c - \sin. b \cos. c \cos. A; \\ \sin. a \cos. C &= \cos. c \sin. b - \sin. c \cos. b \cos. A; \\ \sin. b \cos. C &= \cos. c \sin. a - \sin. c \cos. a \cos. B; \\ \sin. b \cos. A &= \cos. a \sin. c - \sin. a \cos. c \cos. B; \\ \sin. c \cos. A &= \cos. a \sin. b - \sin. a \cos. b \cos. C; \\ \sin. c \cos. B &= \cos. b \sin. a - \sin. b \cos. a \cos. C. \end{aligned} \right\} \quad (I.)$$

If we apply the principle of the Polar Triangle to equations (I) they become

$$\left. \begin{aligned} \sin. A \cos. b &= \cos. B \sin. C + \sin. B \cos. C \cos. a; \\ \sin. A \cos. c &= \cos. C \sin. B + \sin. C \cos. B \cos. a; \\ \sin. B \cos. c &= \cos. C \sin. A + \sin. C \cos. A \cos. b; \\ \sin. B \cos. a &= \cos. A \sin. C + \sin. A \cos. C \cos. b; \\ \sin. C \cos. a &= \cos. A \sin. B + \sin. A \cos. B \cos. c; \\ \sin. C \cos. b &= \cos. B \sin. A + \sin. B \cos. A \cos. c. \end{aligned} \right\} \quad (J.)$$

CHAPTER VI.

SOLUTION OF SPHERICAL RIGHT TRIANGLES.

§ 65. We shall confine ourselves to such triangles as have only one right angle; those that have two or three right angles will be considered hereafter. (See § 74.)

A spherical triangle consists of 6 parts, 3 sides and 3 angles; and any three of these being known the others may be found. In the present case, one of the angles being a right angle, it follows that any other two parts being known the other three may be found.

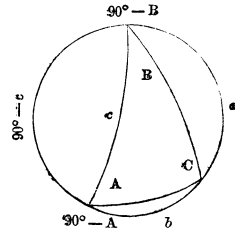
The number of combinations of 5 things taken 3 and 3 at a time, is $\frac{5 \times 4 \times 3}{1 \times 2 \times 3} = 10$; therefore, ten different cases present themselves in the solution of right-angled spherical triangles.

§ 66. The solution of these ten cases may all be comprised in two rules first given by *Napier*, and known under the name of *Napier's Rules for Circular Parts*.

OF THE CIRCULAR PARTS.

The right angle is not taken into consideration. The two sides, the complements of the two angles, and the complement of the hypotenuse constitute the five circular parts.

Let ABC be a spherical triangle, right-angled at C . If we arrange the five circulating parts upon the circumference of its circumscribed circle, we shall readily discover which are *adjacent*, and which are *opposite*, when either of the five parts is chosen as the *middle part*.



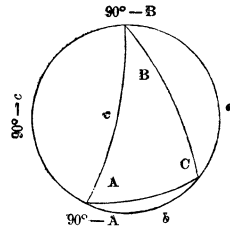
NAPIER'S RULES.

I. *The sine of the middle part is equal to the product of the tangents of the adjacent parts.*

II. *The sine of the middle part is equal to the product of the cosines of the opposite parts.*

If now we take in turn each of the five parts as the middle part, and apply these Rules, we shall obtain ten formulas, as follows.

First. Let a be the middle part, then will b and $(90^\circ - B)$ be the adjacent parts, and $(90^\circ - c)$ and $(90^\circ - A)$ the opposite parts.



$$\text{RULE I. } \sin. a = \tan. b \tan. (90^\circ - B) = \tan. b \cot. B. \quad (1.)$$

$$\text{RULE II. } \sin. a = \cos. (90^\circ - c) \cos. (90^\circ - A) = \sin. c \sin. A. \quad (2.)$$

Secondly. Let b be the middle part, then will a and $(90^\circ - A)$ be the adjacent parts, and $(90^\circ - c)$ and $(90^\circ - B)$ the opposite parts.

$$\text{RULE I. } \sin. b = \tan. a \tan. (90^\circ - A) = \tan. a \cot. A. \quad (3.)$$

$$\text{RULE II. } \sin. b = \cos. (90^\circ - c) \cos. (90^\circ - B) = \sin. c \sin. B. \quad (4.)$$

Hence, (1), (2), (5), (6,) (7), and (8) give all that is required for the solution of all spherical right triangles.

That these six formulas, as drawn from *Napier's Rules*, are correct, may be shown as follows :

Equation (4) of (H) § 62, gives, when $C = 90^\circ$,

$$\begin{aligned} \cot. B &= \cot. b \sin. a, \text{ which immediately reduces to} \\ \sin. a &= \tan. b \cot. B, \text{ which is (1) of (K).} \end{aligned}$$

Equation (2) of (E) § 59, gives, when $C = 90^\circ$,

$$\sin. A = \frac{\sin. a}{\sin. c}, \text{ or } \sin. a = \sin. c \sin. A, \text{ which is (2) of (K).}$$

Equation (3) of (B) § 56, gives, when $C = 90^\circ$,

$$\cos. c = \frac{\cos. A \cos. B}{\sin. A \sin. B}, \text{ or } \cos. c = \cot. A \cot. B, \text{ which is (5) of (K).}$$

Equation (3) of (A) § 55, gives, when $C = 90^\circ$,

$$0 = \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b}, \text{ or } \cos. c = \cos. a \cos. b, \text{ which is (6) of (K).}$$

Equation (6) of (H) § 62, gives, when $C = 90^\circ$,

$$\begin{aligned} 0 &= \cot. c \sin. b \operatorname{cosec}. A - \cos. b \cot. A, \text{ or } \cos. A = \tan. b \cot. c, \\ &\text{which is (7) of (K).} \end{aligned}$$

Equation (1) of (B) § 56, gives, when $C = 90^\circ$,

$$\cos. a = \frac{\cos. A}{\sin. B}, \text{ or } \cos. A = \cos. a \sin. B, \text{ which is (8) of (K).}$$

Having proved the formulas (K), which were derived from *Napier's two Rules*, we are at liberty to use them in solving the six cases of spherical right triangles, or we may in all cases make the direct application of those Rules.

§ 67. That we may, in the foregoing formulas, distinguish the trigonometric functions of parts less than 90° from those greater than 90° , we must pay particular attention to the rule for the algebraic signs.

Since the sine of an angle and the sine of its supplement are identical, it follows that there will be two solutions when the required part is determined by means of its sine, unless the ambiguity can be removed by the application of the following propositions :

PROPOSITION I.

In a spheric right triangle, an angle and its opposite side are always both less than 90° , or else both greater than 90° .

To establish this, take equation (8) of (K),

$$\cos. A = \cos. a \sin. B,$$

which immediately gives

$$\sin B = \frac{\cos. A}{\cos. a}. \quad . \quad . \quad . \quad (1.)$$

Now, since B is less than 180° , its sine must be positive, hence the fraction which is equal to $\sin. B$ must also be positive; therefore the numerator and denominator must be either both positive or else both negative—that is, A and a must be either both less than 90° , or else both greater than 90° .

PROPOSITION II.

When the two sides of a spherical triangle including the right angle are both less or both greater than 90° , the hypotenuse will be less than 90° . But when one side is less than 90° and the other greater than 90° , the hypotenuse will be greater than 90° .

This may be established by equation (6) of (K), which is

$$\cos. c = \cos. a \cos. b.$$

If a and b are both less than 90° , their cosines will be positive; if they are both greater than 90° , their cosines will be negative, and in both cases the product of their cosines, which gives $\cos. c$, will be positive; consequently c will be less than 90° .

If a and b are in different quadrants—that is, if one is less than 90° and the other greater than 90° , the product of their cosines will be negative; consequently c will be greater than 90° .

CASE I.

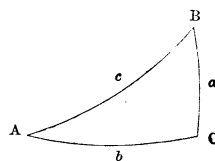
§ 68. *Given the hypotenuse and one angle.*

For example, suppose c and A given.

To find a . Equation (2) of (K) immediately gives

$$\sin. a = \sin. c \sin A.$$

There will be two values of a correspond-



ing to the same sine. We must, § 67, use the one which is in the same quadrant as A (Proposition I.).

To find b . Equation (7) of (K) readily gives

$$\tan. b = \tan. c \cos. A.$$

To find B . Equation (5) of (K) readily gives

$$\cot. B = \cos. c \tan. A.$$

The quadrants in which b and B are situated are determined by the algebraic sign of $\tan. b$ and $\cot. B$, § 67.

EXAMPLES.

1. Given $c = 104^\circ 30'$, $A = 75^\circ 15'$. To solve the triangle,

$$\log. \sin. c = 9.985942$$

$$\log. \sin. A = 9.985447$$

$$\log. \sin. a = 9.971389$$

Therefore $a = 69^\circ 25' 48''$, or $110^\circ 34' 12''$.

By Prop. I., under § 67, it follows that a must be in the same quadrant with A ; consequently we must take $69^\circ 25' 48''$ as the value of a . The supplement of this is not applicable to this triangle.

$$\log. \tan. c = 10.587342n$$

$$\log. \cos. c = 9.398600n$$

$$\log. \cos. A = 9.405862$$

$$\log. \tan. A = 10.579585$$

$$\log. \tan. b = 9.993204n$$

$$\log. \cot. B = 9.978185n$$

$$b = 135^\circ 26' 54''.$$

$$B = 133^\circ 33' 42''.$$

In the above work, since c is an arc greater than a quadrant, its tangent and cosine are both negative, hence at the right of their logarithms we have placed the letter n . (See Note under Case IV., § 50.)

Now, since the addition of logarithms corresponds to multiplication, and the subtraction of one logarithm from another corresponds to division, it follows that when both logarithms, in the case of addition or of subtraction, are derived from negative values—that is, when both logarithms have the letter n —the result must be positive, and its logarithm will not be marked with n . But when one of the logarithms is distinguished with n and the other is not, the resulting logarithm is also to be marked with n , and the corresponding numerical value will be negative.

Our results for the logarithms of $\tan. b$ and $\cot. B$ above are both marked with n , consequently their numerical values are both negative. We seek in the Table in the usual manner for the arcs whose logarithmic tangent and cotangent are respectively 9.993204 and 9.978185, and find them to be $44^\circ 33' 6''$ and $46^\circ 26' 18''$; we then take the supplements of those arcs for the values sought. Or, we may at

once take out of the Tables the obtuse angles, since our Tables have been so arranged as to give the supplementary angles in all cases. By this arrangement, the logarithmic work with obtuse angles is performed with the same ease as in the case of acute angles.

2. Given $c = 65^\circ 5'$, $A = 48^\circ 12'$, to solve the triangle.

$$\text{Ans. } \begin{cases} a = 42^\circ 32' 20'' \\ b = 55^\circ 7' 32'' \\ B = 64^\circ 46' 13'' \end{cases}$$

CASE II.

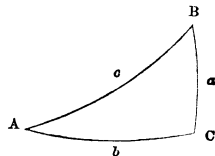
§ 69. *Given the hypotenuse and a side:* for example, c and a .

To find b. Equation (6) of (K) gives

$$\cos. b = \frac{\cos. c}{\cos. a}.$$

To find A. Equation (2) of (K) gives

$$\sin. A = \frac{\sin. a}{\sin. c}.$$



We must take A in the same quadrant as a , § 67, Prop. I.

To find B. Equation (9) of (K) is

$$\cos. B = \tan. a \cot. c.$$

EXAMPLES.

1. Given $c = 94^\circ 5'$, $a = 100^\circ 45'$, to solve the triangle.

$\cos. c = 8.852525 n$	$\sin. a = 9.992311$	$\tan. a = 10.721576 n$
$\cos. a = 9.270735 n$	$\sin. c = 9.998896$	$\cot. c = 8.853628 n$
$\cos. b = 9.581790$	$\sin. A = 9.993415$	$\cos. B = 9.575204$
$b = 67^\circ 33' 26'',$	$A = 99^\circ 57' 8'',$	$B = 67^\circ 54' 47''.$

In this example, since c and a are each greater than 90° , their cosines, as used in finding the side b , are negative; this we have indicated by the letter n at the right of their logarithms. (See Note under Case IV., § 50.) Also their tangents as well as cotangents are negative, and the letter n indicates this in the work for finding B .

Since the sines of all angles not exceeding 180° are positive, it follows that the logarithms of the sines of a and c , as used in finding A , do not require to be distinguished by the letter n . But the value of A being given in a sine leads to two arcs $80^\circ 2' 52'', 99^\circ 57' 8''$, supplements of each other. Prop. I., § 67, requires us to use the obtuse angle for the value of A .

2. Given $c = 66^\circ 32'$, $a = 37^\circ 48'$, to solve the triangle.

$$\text{Ans. } \begin{cases} b = 59^\circ 44' 13'' \\ A = 41^\circ 55' 34'' \\ B = 70^\circ 18' 42'' \end{cases}$$

CASE III.

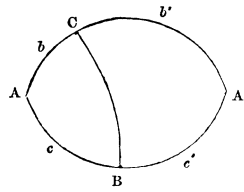
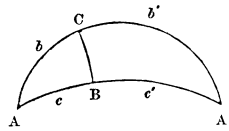
§ 70. *Given one side and its opposite angle:* for example, a and A .

Equations (3), (2), and (8) of (K) readily give

$$\sin. b = \tan. a \cot. A,$$

$$\sin. c = \frac{\sin. a}{\sin. A}.$$

$$\sin. B = \frac{\cos. A}{\cos. a}.$$



In this case there are always two solutions. And we also remark, see Proposition I. under § 67, that a and A are both less than 90° , as in firstfigure, or else both greater than 90° , as in second figure.

If AB and AC be produced to meet at A' , we shall have ABA' and ACA' each a semi-circumference, and the angle $A = A'$. The two triangles ABC , $A'BC$ both have the same given parts a and $A = A'$, but the required parts, b' , c' , B' , of the second triangle are respectively the supplements of b , c , and B of the first triangle.

EXAMPLES.

1. Given $a = 110^\circ 4'$, $A = 98^\circ 30'$, to solve the triangle.

$\tan. a = 10.437364n$	$\sin. a = 9.972802$	$\cos. A = 9.169702n$
$\cot. A = 9.174499n$	$\sin. A = 9.995203$	$\cos. a = 9.535438n$
$\sin. b = 9.611863$	$\sin. c = 9.977599$	$\sin. B = 9.634264$

$$b = \begin{cases} 24^\circ 9' 1'', \\ 155^\circ 50' 59''. \end{cases} \quad c = \begin{cases} 71^\circ 45' 19'', \\ 108^\circ 14' 41''. \end{cases} \quad B = \begin{cases} 25^\circ 31' 3'', \\ 154^\circ 28' 57''. \end{cases}$$

By paying attention to the Propositions under § 67, we shall have for the two solutions as follows:

$$\text{Ans. } \left\{ \begin{array}{l} b = 24^\circ 9' 1'' \\ c = 108^\circ 14' 41'' \\ B = 25^\circ 31' 3'' \end{array} \right\} \text{ or } \left\{ \begin{array}{l} b = 155^\circ 50' 59'' \\ c = 71^\circ 45' 19'' \\ B = 154^\circ 28' 57'' \end{array} \right\}$$

2. Given $a = 50^\circ 12'$, $A = 75^\circ 30'$, to solve the triangle.

$$\text{Ans. } \left\{ \begin{array}{l} b = 18^\circ 5' 0'' \\ c = 52^\circ 31' 10'' \\ B = 23^\circ 1' 35'' \end{array} \right\} \text{ or } \left\{ \begin{array}{l} b = 161^\circ 55' 0'' \\ c = 127^\circ 28' 50'' \\ B = 156^\circ 58' 25'' \end{array} \right\}$$

CASE IV.

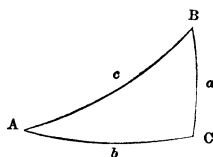
§ 71. *Given one side and its adjacent angle:* for example, a and B .

Equations (1), (9), and (8) of (K) readily give

$$\tan. b = \sin. a \tan. B,$$

$$\cot. c = \cot. a \cos. B,$$

$$\cos. A = \cos. a \sin. B.$$



EXAMPLES.

1. Given $a = 101^\circ 30'$, $B = 42^\circ 15'$, to solve the triangle.

$\sin. a = 9.991193$	$\cot. a = 9.308463 n$	$\cos. a = 9.299655 n$
$\tan. B = 9.958247$	$\cos. B = 9.869360$	$\sin. B = 9.827606$
$\tan. b = 9.949440$	$\cot. c = 9.177823 n$	$\cos. A = 9.127261 n$
$b = 41^\circ 40' 21'',$	$c = 98^\circ 33' 52'',$	$A = 97^\circ 42' 13''.$

Since the logarithms of $\cot. c$ and $\cos. A$ have the letter n , indicating that the $\cot. c$ and $\cos. A$ are negative (see remarks under Case II.), it follows that we must take from the Tables the arcs which are greater than 90° . We thus find

$$\text{Ans. } \left\{ \begin{array}{l} b = 41^\circ 40' 21''. \\ c = 98^\circ 33' 52''. \\ A = 97^\circ 42' 13''. \end{array} \right.$$

2. Given $a = 48^\circ 30'$, $B = 40^\circ 20'$, to solve the triangle.

$$\text{Ans. } \left\{ \begin{array}{l} b = 32^\circ 27' 10''. \\ c = 56^\circ 0' 13''. \\ A = 64^\circ 36' 15''. \end{array} \right.$$

CASE V.

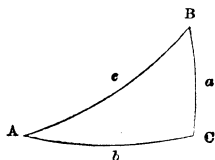
§ 72. *Given the two sides, a and b .*

Equations (6), (3), and (1) of (K) readily give

$$\cos. c = \cos. a \cos. b,$$

$$\cot. A = \cot. a \sin. b,$$

$$\cot. B = \sin. a \cot. b.$$



EXAMPLES.

1. Given $a = 75^\circ 15'$, $b = 120^\circ 15'$, to solve the triangle.

$\cos. a = 9.405862$	$\cot. a = 9.420415$	$\sin. a = 9.985447$
$\cos. b = 9.702236 n$	$\sin. b = 9.936431$	$\cot. b = 9.765805 n$
$\cos. c = 9.108098 n$	$\cot. A = 9.356846$	$\cot. B = 9.751252 n$
$c = 97^\circ 22' 9''$,	$A = 77^\circ 11' 14''$,	$B = 119^\circ 25' 17''$.

2. Given $a = 120^\circ$, $b = 30^\circ$, to solve the triangle.

$$Ans. \begin{cases} c = 115^\circ 39' 32'' \\ A = 106^\circ 6' 8'' \\ B = 33^\circ 42' 36'' \end{cases}$$

CASE VI.

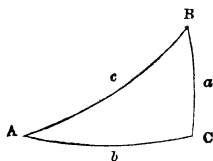
§ 73. *Given the two angles, A and B .*

Equations (8), (10), and (5) of (K) readily give

$$\cos. a = \frac{\cos. A}{\sin. B},$$

$$\cos. b = \frac{\cos. B}{\sin. A},$$

$$\cos. c = \cot. A \cot. B.$$



EXAMPLES.

1. Given $A = 62^\circ 15'$, $B = 56^\circ 30'$, to solve the triangle.

$\cos. A = 9.668027$	$\cos. B = 9.741889$	$\cot. A = 9.721089$
$\sin. B = 9.921107$	$\sin. A = 9.946937$	$\cot. B = 9.820783$
$\cos. a = 9.746920$	$\cos. b = 9.794952$	$\cos. c = 9.541872$
$a = 56^\circ 3' 25''$,	$b = 51^\circ 24' 56''$,	$c = 69^\circ 37' 14''$.

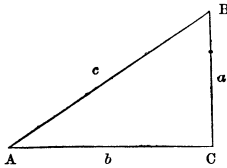
2. Given $A = 99^\circ 30'$, $B = 150^\circ 15'$, to solve the triangle.

$$\text{Ans. } \begin{cases} a = 109^\circ 25' 39'' \\ b = 151^\circ 40' 30'' \\ c = 72^\circ 58' 30'' \end{cases}$$

§ 74. When a spherical triangle has two or even three of its angles right, it is then isosceles, and may be divided into two equal right triangles, each having only one right angle (except the unique case of three right angles in the given triangle), and the solution may be obtained by the foregoing cases. When all the three angles are right, the sides are all equal, and each a quadrant.

§ 75. The foregoing relations of spheric right triangles may be associated with the corresponding relations of plane right triangles as follows:

In plane right triangles.



$$\sin A = \frac{a}{c}; \sin B = \frac{b}{c}$$

$$\cos A = \frac{b}{c}; \cos B = \frac{a}{c}$$

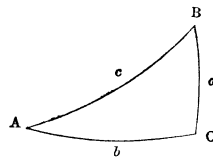
$$\tan A = \frac{a}{b}; \tan B = \frac{b}{a}$$

$$\sin A = \cos B; \sin B = \cos A$$

$$c^2 = a^2 + b^2$$

$$1 = \cot A \cot B$$

In spheric right triangles.



$$\sin A = \frac{\sin a}{\sin c}; \sin B = \frac{\sin b}{\sin c}$$

$$\cos A = \frac{\tan b}{\tan c}; \cos B = \frac{\tan a}{\tan c}$$

$$\tan A = \frac{\tan a}{\sin b}; \tan B = \frac{\tan b}{\sin a}$$

$$\sin A = \frac{\cos B}{\cos b}; \sin B = \frac{\cos A}{\cos a}$$

$$\left\{ \begin{array}{l} \cos c = \cos a \cos b, \text{ or} \\ \log \cos c = \log \cos a + \log \cos b \end{array} \right\}$$

$$\cos c = \cot A \cot B$$

CHAPTER VII.

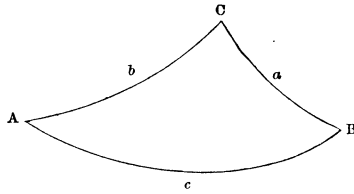
SOLUTION OF SPHERICAL OBLIQUE TRIANGLES.

CASE I.

§ 76. *When two sides and the included angle are given.*

For example, suppose a , b , and C to be given.

Using formula (F), and writing $B - A$ for $A - B$, and $b - a$ for $a - b$, since b is greater than a , and consequently $B > A$, the greater angle being opposite the greater side, we have



$$\tan. \frac{1}{2}(B + A) = \frac{\cos. \frac{1}{2}(b - a)}{\cos. \frac{1}{2}(b + a)} \cot. \frac{1}{2} C;$$

$$\tan. \frac{1}{2}(B - A) = \frac{\sin. \frac{1}{2}(b - a)}{\sin. \frac{1}{2}(b + a)} \cot. \frac{1}{2} C.$$

Knowing the tangents of $\frac{1}{2}(B + A)$ and $\frac{1}{2}(B - A)$, we readily find from the Tables $\frac{1}{2}(B + A)$ and $\frac{1}{2}(B - A)$, and consequently, by addition and subtraction, we have B and A .

The first equation of (G) gives

$$\tan. \frac{1}{2} c = \frac{\cos. \frac{1}{2}(B + A)}{\cos. \frac{1}{2}(B - A)} \tan. \frac{1}{2}(b + a).$$

And, in like manner, if any two sides and the included angle be given, the remaining parts may be found.

EXAMPLES.

1. Given $a = 70^\circ 35'$; $b = 120^\circ 15'$; $C = 54^\circ 40'$, to find the remaining parts.

$$\frac{1}{2}(b + a) = 95^\circ 25'; \quad \frac{1}{2}(b - a) = 24^\circ 50'; \quad \frac{1}{2} C = 27^\circ 20'.$$

$$\begin{array}{lll} \cos. \frac{1}{2}(b - a) = 9.957863 & \sin. \frac{1}{2}(b - a) = 9.623229 & \\ \text{ar. co. cos. } \frac{1}{2}(b + a) = 1.025038n & \text{ar. co. sin. } \frac{1}{2}(b + a) = 0.001944 & \\ \cot. \frac{1}{2} C = 10.286614 & . & . & . & 10.286614 \end{array}$$

$$\tan. \frac{1}{2}(B + A) = 11.269515n \qquad \tan. \frac{1}{2}(B - A) = 9.911787$$

$$\frac{1}{2}(B + A) = 93^\circ 4' 39''. \qquad \frac{1}{2}(B - A) = 39^\circ 13' 14''.$$

Consequently, $A = 53^\circ 51' 25''$; $B = 132^\circ 17' 53''$, observing that the greater angle is opposite the greater side (Geom., B. VIII., T. XIV.).

$$\begin{aligned}\cos. \frac{1}{2}(B+A) &= 8.729865n \\ \text{ar. co. cos. } \frac{1}{2}(B-A) &= 0.110856 \\ \tan. \frac{1}{2}(b+a) &= 11.023094n \\ \tan. \frac{1}{2}c &= 9.863815\end{aligned}$$

$$\frac{1}{2}c = 36^\circ 9' 37'', \text{ and } c = 72^\circ 19' 14''.$$

Collecting results, we have

$$\text{Ans. } \begin{cases} A = 53^\circ 51' 25''. \\ B = 132^\circ 17' 53''. \\ c = 72^\circ 19' 14''. \end{cases}$$

In this case there can be no ambiguity—only one solution can be obtained.

2. Given $a = 100^\circ 50'$; $b = 99^\circ 30'$; $C = 68^\circ 40'$, to find the remaining parts.

$$\text{Ans. } \begin{cases} A = 97^\circ 51' 59''. \\ B = 95^\circ 52' 59''. \\ c = 67^\circ 27' 10''. \end{cases}$$

CASE II.

§ 77. When two angles and the interjacent side are given.

For example, suppose A , B , c to be given.

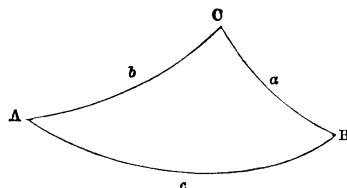
By formula (G) we have

$$\begin{aligned}\tan. \frac{1}{2}(b+a) &= \frac{\cos. \frac{1}{2}(B-A)}{\cos. \frac{1}{2}(B+A)} \tan. \frac{1}{2}c, \\ \tan. \frac{1}{2}(b-a) &= \frac{\sin. \frac{1}{2}(B-A)}{\sin. \frac{1}{2}(B+A)} \tan. \frac{1}{2}c.\end{aligned}$$

Having found a and b by the above, we use the first equation of formula (F), which readily gives

$$\cot. \frac{1}{2}C = \frac{\cos. \frac{1}{2}(b+a)}{\cos. \frac{1}{2}(b-a)} \tan. \frac{1}{2}(B+A).$$

And, in like manner, if any two angles and the interjacent side be given, the remaining parts may be found.



EXAMPLES.

1. Given $c = 110^\circ$; $A = 84^\circ$; $B = 88^\circ$, to determine the remaining parts.

$$\frac{1}{2}(B + A) = 86^\circ; \frac{1}{2}(B - A) = 2^\circ; \frac{1}{2}c = 55^\circ.$$

$$\begin{array}{ll} \cos. \frac{1}{2}(B - A) = 9.999735 & \sin. \frac{1}{2}(B - A) = 8.542819 \\ \text{ar. co. cos. } \frac{1}{2}(B + A) = 1.156415 & \text{ar. co. sin. } \frac{1}{2}(B + A) = 0.001059 \\ \tan. \frac{1}{2}c = 10.154773 & . \quad . \quad . \quad . = 10.154773 \\ \tan. \frac{1}{2}(b + a) = 11.310923 & \tan. \frac{1}{2}(b - a) = 8.698651 \end{array}$$

$$\frac{1}{2}(b + a) = 87^\circ 12' 7''.$$

$$\frac{1}{2}(b - a) = 2^\circ 51' 37''.$$

Consequently, $a = 84^\circ 20' 30''$; $b = 90^\circ 3' 44''$.

Observing that the greater side is opposite the greater angle (Geom., B. VIII., T. XIV.)

$$\begin{array}{ll} \cos. \frac{1}{2}(b + a) = 8.688563 \\ \text{ar. co. cos. } \frac{1}{2}(b - a) = 0.000541 \\ \tan. \frac{1}{2}(B + A) = 11.155356 \\ \cot. \frac{1}{2}C = 9.844460 \end{array}$$

$$\frac{1}{2}C = 55^\circ 2' 51'', \text{ and } C = 110^\circ 5' 42''.$$

Collecting, we have

$$\text{Ans. } \begin{cases} a = 84^\circ 20' 30''. \\ b = 90^\circ 3' 44''. \\ C = 110^\circ 5' 42''. \end{cases}$$

This case can never admit of more than one solution.

2. Given $c = 70^\circ$; $A = 140^\circ$; $B = 50^\circ$, to find the other parts.

$$\text{Ans. } \begin{cases} a = 126^\circ 24' 41''. \\ b = 73^\circ 33' 19''. \\ C = 48^\circ 38' 18''. \end{cases}$$

CASE III.

§ 78. When two sides and angle opposite one of them are given.

For example, suppose a , b , and A to be given.

We first find B by (E),

$$\frac{\sin. A}{\sin. B} = \frac{\sin. a}{\sin. b},$$

which gives

$$\sin. B = \frac{\sin. b \sin. A}{\sin. a}. \quad (1.)$$

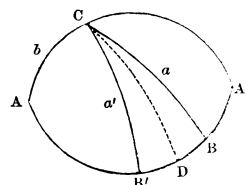
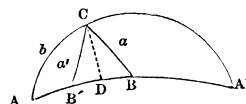
Having found B , we proceed to find C and c by means of the following, drawn from (F) and (G):

$$\cot. \frac{1}{2} C = \frac{\cos. \frac{1}{2} (a + b) \tan. \frac{1}{2} (A + B)}{\cos. \frac{1}{2} (a - b)}. \quad (2.)$$

$$\tan. \frac{1}{2} c = \frac{\cos. \frac{1}{2} (A + B) \tan. \frac{1}{2} (a + b)}{\cos. \frac{1}{2} (A - B)}. \quad (3.)$$

Equation (1) will give two values for B supplementary to each other. If these two values of B , when used in (2) and (3), do not cause either to become negative, there will be two solutions. Since the arcs C and c are each less than 180° , $\frac{1}{2} C$ and $\frac{1}{2} c$ must each be less than 90° , and their tangents and cotangents must be positive. Hence their values given by the right-hand members of (2) and (3) must also be positive.

If we draw the perpendicular CD , we see that it will be the shortest arc that can be drawn from C , in case the angle A (see first figure) is acute; but this perpendicular will be the longest arc that can be drawn from C , when the angle A is obtuse (see second figure). From this we see that a may frequently take two positions, as indicated in the diagrams, on opposites sides of this perpendicular, thus giving two triangles ABC , $AB'C$, both fulfilling the required conditions.



EXAMPLES.

1. Given $a = 40^\circ$; $b = 118^\circ$; $A = 30^\circ$, to solve the triangle.

$$\sin. b = 9.945935$$

$$\text{ar. co. sin. } a = 0.191933$$

$$\sin. A = 9.698970$$

$$\sin. B = 9.836838$$

$$B = 43^\circ 22' 42'', \text{ or } 136^\circ 37' 18''.$$

$$B = 43^\circ 22' 42'', \text{ first value of } B.$$

$$A = 30^\circ.$$

$$\frac{1}{2}(B + A) = 36^\circ 41' 21''; \quad \frac{1}{2}(B - A) = 6^\circ 41' 21''.$$

$$\frac{1}{2}(b + a) = 79^\circ; \quad \frac{1}{2}(b - a) = 39^\circ.$$

$$\cos. \frac{1}{2}(b + a) = 9.280599$$

$$\cos. \frac{1}{2}(B + A) = 9.904114$$

$$\text{ar. co. cos. } \frac{1}{2}(b - a) = 0.109497$$

$$\text{ar. co. cos. } \frac{1}{2}(B - A) = 0.002966$$

$$\tan. \frac{1}{2}(B + A) = 9.872204$$

$$\tan. \frac{1}{2}(b + a) = 10.711348$$

$$\cot. \frac{1}{2} C = 9.262300$$

$$\tan. \frac{1}{2} c = 10.618428$$

$$\frac{1}{2} C = 79^\circ 37' 59'',$$

$$\frac{1}{2} c = 76^\circ 27' 48'',$$

$$\text{and } C = 159^\circ 15' 58'';$$

$$\text{and } c = 152^\circ 55' 36''.$$

$$B = 136^\circ 37' 18'', \text{ second value of } B.$$

$$A = 30^\circ.$$

$$\frac{1}{2}(B + A) = 83^\circ 18' 39''; \quad \frac{1}{2}(B - A) = 53^\circ 18' 39''.$$

$$\cos. \frac{1}{2}(b + a) = 9.280599$$

$$\cos. \frac{1}{2}(B + A) = 9.066264$$

$$\text{ar. co. cos. } \frac{1}{2}(b - a) = 0.109497$$

$$\text{ar. co. cos. } \frac{1}{2}(B - A) = 0.223681$$

$$\tan. \frac{1}{2}(B + A) = 10.930770$$

$$\tan. \frac{1}{2}(b + a) = 10.711348$$

$$\cot. \frac{1}{2} C = 10.320866$$

$$\tan. \frac{1}{2} c = 10.001293$$

$$\frac{1}{2} C = 25^\circ 31' 58'',$$

$$\frac{1}{2} c = 45^\circ 5' 7'',$$

$$\text{and } C = 51^\circ 3' 56'';$$

$$\text{and } c = 90^\circ 10' 14''.$$

Collecting results, we have these two solutions:

$$\text{Ans. } \left\{ \begin{array}{l} B = 43^\circ 22' 42'' \\ C = 159^\circ 15' 58'' \\ c = 152^\circ 55' 36'' \end{array} \right\} \text{ or } \left\{ \begin{array}{l} B = 136^\circ 37' 18'' \\ C = 51^\circ 3' 56'' \\ c = 90^\circ 10' 14'' \end{array} \right.$$

2. Given $a = 100^\circ$; $b = 64^\circ 30'$; $A = 95^\circ 30'$, to find the other parts.

There is only one solution, as follows:

$$\text{Ans. } \begin{cases} B = 65^\circ 49' 25'' \\ C = 98^\circ 32' 24'' \\ c = 101^\circ 55' 58'' \end{cases}$$

CASE IV.

§ 79. When two angles and the side opposite one of them are given.

For example, suppose A , B , and a to be given.

We first find b by (E),

$$\frac{\sin. A}{\sin. B} = \frac{\sin. a}{\sin. b},$$

which gives

$$\sin. b = \frac{\sin. a \sin. B}{\sin. A}. \quad (1.)$$

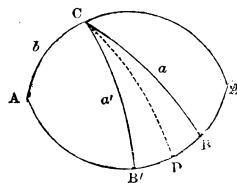
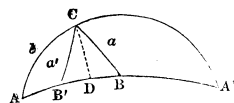
Having found b , we proceed to find C and c , by using precisely the same formulas as in Case III., which are

$$\cot. \frac{1}{2} C = \frac{\cos. \frac{1}{2} (a + b) \tan. \frac{1}{2} (A + B)}{\cos. \frac{1}{2} (a - b)}. \quad (2.)$$

$$\tan. \frac{1}{2} c = \frac{\cos. \frac{1}{2} (A + B) \tan. \frac{1}{2} (a + b)}{\cos. \frac{1}{2} (A - B)}. \quad (3.)$$

As in the last case, equation (1) will give two values for b supplementary to each other.

If these two values of b , when used in (2) and (3), do not cause either to become negative, there will be two solutions. These two triangles, which thus satisfy the conditions of the problem, will be indicated in the above diagrams by ABC , $A'B'C$.



EXAMPLES.

1. Given $A = 140^\circ$; $B = 60^\circ$; $a = 150^\circ$, to solve the triangle.

$$\sin. a = 9.698970$$

$$\text{ar. co. sin. } A = 0.191933$$

$$\sin. B = 9.937531$$

$$\sin. b = 9.828434$$

$$b = 42^\circ 20' 58'' \text{ or } 137^\circ 39' 2''.$$

$$a = 150^\circ$$

$$b = 42^\circ 20' 58'', \text{ first value of } b.$$

$$\frac{1}{2}(a + b) = 96^\circ 10' 29''; \frac{1}{2}(a - b) = 53^\circ 49' 31''.$$

$$\frac{1}{2}(A + B) = 100^\circ; \frac{1}{2}(A - B) = 40^\circ.$$

$$\cos. \frac{1}{2}(a + b) = 9.031654n$$

$$\cos. \frac{1}{2}(A + B) = 9.239670n$$

$$\text{ar.co.cos. } \frac{1}{2}(a - b) = 0.228964$$

$$\text{ar.co.cos. } \frac{1}{2}(A - B) = 0.115746$$

$$\tan. \frac{1}{2}(A + B) = 10.753681n$$

$$\tan. \frac{1}{2}(a + b) = 10.965820n$$

$$\cot. \frac{1}{2}C = 10.014299$$

$$\tan. \frac{1}{2}c = 10.321236$$

$$\frac{1}{2}C = 44^\circ 3' 25'',$$

$$\frac{1}{2}c = 64^\circ 29' 11'',$$

$$\text{and } C = 88^\circ 6' 50''.$$

$$\text{and } c = 128^\circ 58' 22''.$$

$$a = 150^\circ$$

$$b = 137^\circ 39' 2'', \text{ second value of } b.$$

$$\frac{1}{2}(a + b) = 143^\circ 49' 31''; \frac{1}{2}(a - b) = 6^\circ 10' 29''.$$

$$\cos. \frac{1}{2}(a + b) = 9.906993n$$

$$\cos. \frac{1}{2}(A + B) = 9.239670n$$

$$\text{ar.co.cos. } \frac{1}{2}(a - b) = 0.002525$$

$$\text{ar.co.cos. } \frac{1}{2}(A - B) = 0.115746$$

$$\tan. \frac{1}{2}(A + B) = 10.753681n$$

$$\tan. \frac{1}{2}(a + b) = 9.864043n$$

$$\cot. \frac{1}{2}C = 10.663199$$

$$\tan. \frac{1}{2}c = 9.219459$$

$$\frac{1}{2}C = 12^\circ 15' 10'',$$

$$\frac{1}{2}c = 9^\circ 24' 41'',$$

$$\text{and } C = 24^\circ 30' 20''.$$

$$\text{and } c = 18^\circ 49' 22''.$$

Collecting results, we have these two solutions:

$$\text{Ans. } \left\{ \begin{array}{l} b = 42^\circ 20' 58'' \\ c = 128^\circ 58' 22'' \\ C = 88^\circ 6' 50'' \end{array} \right\} \text{ or } \left\{ \begin{array}{l} b = 137^\circ 39' 2'' \\ c = 18^\circ 49' 22'' \\ C = 24^\circ 30' 20'' \end{array} \right\}.$$

2. Given $A = 80^\circ 20'$; $B = 115^\circ 30'$; $a = 84^\circ 20'$, to find the other parts.

There is only one solution, as follows :

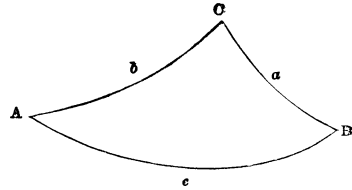
$$\text{Ans. } \begin{cases} b = 114^\circ 20' 26'' \\ c = 82^\circ 39' 34'' \\ C = 79^\circ 17' 8'' \end{cases}$$

CASE V.

§ 80. When the three sides are given.

Either of the groups (C'), (C'') will give the angles.

If we use (C''') we have



$$\tan. \frac{1}{2} A = \left\{ \frac{\sin. (s-b) \sin. (s-c)}{\sin. s \sin. (s-a)} \right\}^{\frac{1}{2}}; \quad . \quad . \quad (1.)$$

$$\tan. \frac{1}{2} B = \left\{ \frac{\sin. (s-c) \sin. (s-a)}{\sin. s \sin. (s-b)} \right\}^{\frac{1}{2}}; \quad . \quad . \quad (2.)$$

$$\tan. \frac{1}{2} C = \left\{ \frac{\sin. (s-a) \sin. (s-b)}{\sin. s \sin. (s-c)} \right\}^{\frac{1}{2}}; \quad . \quad . \quad (3.)$$

where $s = \frac{1}{2} (a + b + c)$.

Since the values of the half angles are given by these formulas, there will be no ambiguity in this case.

EXAMPLES.

1. Given $a = 100^\circ$; $b = 80^\circ$; $c = 75^\circ$, to find the angles.

$$a = 100^\circ$$

$$b = 80^\circ$$

$$c = 75^\circ$$

$$\frac{1}{2} (a + b + c) = s = 127^\circ 30'; \quad s - a = 27^\circ 30'; \quad s - b = 47^\circ 30'; \quad s - c = 52^\circ 30'.$$

$$\begin{array}{ll}
 \sin. (s - b) = 9.867631 & \sin. (s - c) = 9.899467 \\
 \sin. (s - c) = 9.899467 & \sin. (s - a) = 9.664406 \\
 \text{ar. co. sin. } (s - a) = 0.335594 & \text{ar. co. sin. } (s - b) = 0.132369 \\
 \text{ar. co. sin. } s = 0.100533 & \text{ar. co. sin. } s = 0.100533
 \end{array}$$

$$2) 20.203225$$

$$2) 19.796775$$

$$\tan. \frac{1}{2} A = 10.101612$$

$$\tan. \frac{1}{2} B = 9.898387$$

$$\frac{1}{2} A = 38^\circ 21' 27'',$$

$$\frac{1}{2} B = 38^\circ 21' 27'',$$

$$\text{and } A = 76^\circ 42' 54''.$$

$$\text{and } B = 76^\circ 42' 54''.$$

$$\sin. (s - a) = 9.664406$$

$$\sin. (s - b) = 9.867631$$

$$\text{ar. co. sin. } (s - c) = 0.100533$$

$$\text{ar. co. sin. } s = 0.100533$$

$$2) 19.733103$$

$$\tan. \frac{1}{2} C = 9.866551$$

$$\frac{1}{2} C = 36^\circ 19' 57'' \text{ and } C = 72^\circ 39' 54''.$$

2. Given $a = 49^\circ 8'$; $b = 57^\circ 16'$; $c = 96^\circ 12'$, to find the angles.

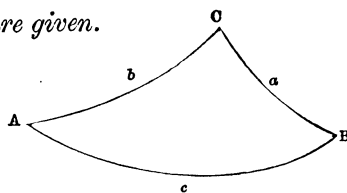
$$\text{Ans. } \begin{cases} A = 31^\circ 32' 42'' \\ B = 35^\circ 35' 15'' \\ C = 136^\circ 32' 48'' \end{cases}$$

CASE VI.

§ 81. When the three angles are given.

As in last case, we have three groups (D'), (D''), (D'''), either of which will give the sides.

If we use (D'''), we have



$$\tan. \frac{1}{2} a = \left\{ \frac{-\cos. S \cos. (S - A)}{\cos. (S - B) \cos. (S - C)} \right\}^{\frac{1}{2}}; \quad \dots (1)$$

$$\tan. \frac{1}{2} b = \left\{ \frac{-\cos. S \cos. (S - B)}{\cos. (S - C) \cos. (S - A)} \right\}^{\frac{1}{2}}; \quad \dots (2)$$

$$\tan. \frac{1}{2} c = \left\{ \frac{-\cos. S \cos. (S - C)}{\cos. (S - A) \cos. (S - B)} \right\}^{\frac{1}{2}}; \quad \dots (3)$$

where $S = \frac{1}{2} (A + B + C)$.

As in last case there can be no ambiguity in the values of a, b, c as here found.

EXAMPLES.

1. Given $A = 110^\circ$; $B = 100^\circ$; $C = 90^\circ$, to find the sides.

$$A = 110^\circ$$

$$B = 100^\circ$$

$$C = 90^\circ$$

$$\frac{1}{2}(A+B+C) = S = 150^\circ; \quad S-A = 40^\circ; \quad S-B = 50^\circ; \quad S-C = 60^\circ.$$

$$\begin{array}{rcll} -\cos. S & = & 9.937531 & = 9.937531 \\ \cos. (S-A) & = & 9.884254 & \cos. (S-B) = 9.808067 \\ \text{ar. co. cos. } (S-B) & = & 0.191933 & \text{ar. co. cos. } (S-C) = 0.301030 \\ \text{ar. co. cos. } (S-C) & = & 0.301030 & \text{ar. co. cos. } (S-A) = 0.115746 \end{array}$$

$$\hline 2) 20.314748$$

$$\hline 2) 20.162374$$

$$\tan. \frac{1}{2} a = 10.157374$$

$$\tan. \frac{1}{2} b = 10.081187$$

$$\begin{array}{l} \frac{1}{2} a = 55^\circ 9' 40'', \\ \text{and } a = 110^\circ 19' 20''. \end{array}$$

$$\begin{array}{l} \frac{1}{2} b = 50^\circ 19' 28'', \\ \text{and } b = 100^\circ 38' 56''. \end{array}$$

$$\begin{array}{rcll} -\cos. S & = & 9.937531 \\ \cos. (S-C) & = & 9.698970 \\ \text{ar. co. cos. } (S-A) & = & 0.115746 \\ \text{ar. co. cos. } (S-B) & = & 0.191933 \end{array}$$

$$\hline 2) 19.944180$$

$$\tan. \frac{1}{2} c = 9.972090$$

$$\frac{1}{2} c = 43^\circ 9' 37'' \text{ and } c = 86^\circ 19' 14''.$$

Since the angle C is right, the solution might have been obtained by Case VI. of Right Triangles. (See Ex. 6, § 82.)

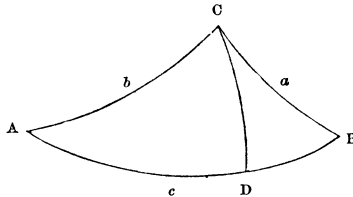
2. Given $A = 85^\circ$; $B = 60^\circ$; $C = 50^\circ$, to find the sides.

$$\text{Ans. } \begin{cases} a = 51^\circ 59' 16''. \\ b = 43^\circ 13' 48''. \\ c = 37^\circ 17' 26''. \end{cases}$$

§ 82. The methods which we have given for the solution of the six cases of spherical oblique triangles, as well as for the six

cases of spherical right triangles, are all direct and always applicable; but frequently other methods may be used for particular cases, which would lead to simpler methods of solution. We can, however, hardly expect any thing more simple or more easy to be retained by the memory, in the case of right triangles, than the *Rules of Napier*; but in oblique triangles we can frequently obtain simpler solutions than by the methods already given. We may obviously divide any oblique triangle into two right triangles, and thus make all the cases of oblique triangles depend, for their solution, upon the principles of *Napier's Rules*.

As an example of this method, let us suppose in the triangle ABC, that we have the arcs AB, AC, and the angle A given, which corresponds with Case I. If we draw the perpendicular arc CD, we shall have, by applying *Napier's Rules* to the right triangles ADC, BDC, as follows:



$$\cos. b = \cos. AD \cos. CD. \quad . \quad . \quad . \quad (1.)$$

$$\cos. a = \cos. BD \cos. CD. \quad . \quad . \quad . \quad (2.)$$

Dividing (1) by (2), we have

$$\frac{\cos. AD}{\cos. BD} = \frac{\cos. b}{\cos. a}, \text{ or } \cos. AD : \cos. BD :: \cos. b : \cos. a. \quad (3.)$$

That is, *if from an angle of a spheric oblique triangle an arc be drawn perpendicular to the opposite side, dividing it into two segments, we shall have the cosines of these segments to each other as the cosines of their adjacent sides.*

Returning to the triangle ADC, we have

$$\tan. AD = \tan. b \cos. A. \quad . \quad . \quad . \quad (4.)$$

Thus we know AD, which subtracted from AB = c, gives BD.

The segments are then known, and condition (3) gives at once $\cos. a$.

It is obvious that, by the aid of the great variety of formulas which we have given, the method of solving many of the cases may be varied to almost any extent we please. We will leave

this kind of exercise wholly to the student, remarking that in our general solutions we have given those methods which, under all circumstances, seemed to be the safest and best.

§ 83. EXAMPLES FOR PRACTICE.

1. Given, in a right spherical triangle, $A = 91^\circ 11'$; $B = 111^\circ 11'$, to find the remaining parts.

$$Ans. \begin{cases} a = 91^\circ 16' 8'' \\ b = 111^\circ 48' 43'' \\ c = 89^\circ 32' 28'' \end{cases}$$

2. Given, in a right spheric triangle, $a = 35^\circ 44'$; $A = 37^\circ 28'$, to find the other parts.

$$Ans. \begin{cases} b = 69^\circ 50' 24'' \\ c = 73^\circ 45' 15'' \\ B = 77^\circ 54' 0'' \end{cases} \text{ or } \begin{cases} b = 110^\circ 9' 36'' \\ c = 106^\circ 14' 45'' \\ B = 102^\circ 6' 0'' \end{cases}$$

3. Given, in a spheric oblique triangle, $a = 138^\circ$; $A = 95^\circ$; $C = 104^\circ$, to find the other parts.

$$Ans. \begin{cases} b = 16^\circ 34' 19'' \\ c = 139^\circ 19' 40'' \\ B = 25^\circ 7' 38'' \end{cases} \text{ or } \begin{cases} b = 7^\circ 0' 21'' \\ c = 40^\circ 40' 20'' \\ B = 10^\circ 27' 42'' \end{cases}$$

4. Given $a = 81^\circ 17'$; $b = 114^\circ 3'$; $c = 59^\circ 12'$, to find the angles.

$$Ans. \begin{cases} A = 62^\circ 39' 43'' \\ B = 124^\circ 50' 50'' \\ C = 50^\circ 31' 43'' \end{cases}$$

5. Given, in a right spheric triangle, $a = 118^\circ 54'$; $B = 12^\circ 19'$, to find the other parts.

$$Ans. \begin{cases} b = 10^\circ 49' 17'' \\ c = 118^\circ 20' 20'' \\ A = 95^\circ 55' 2'' \end{cases}$$

6. Given, in a right spheric triangle, $A = 110^\circ$; $B = 100^\circ$, to find the other part. (This is the same as first example under Case VI. of Oblique Spherical Trigonometry.)

$$Ans. \begin{cases} a = 110^\circ 19' 20'' \\ b = 100^\circ 38' 56'' \\ c = 86^\circ 19' 14'' \end{cases}$$

CHAPTER VIII.

MENSURATION OF SURFACES.

§ 84. THE *area* of a surface is found by comparing it with some known and fixed surface, which is taken as the *unit surface*, which unit surface is usually in the form of a *square*; as, a square inch, a square foot, a square yard, &c.

The *superficial* unit receives its name from that of the *linear* unit, which measures its side. Thus, a square whose side is one inch is called a square inch; one whose side is one foot is called a square foot; one whose side is one yard is called a square yard.

There are some superficial units which have no corresponding linear unit, such as the *acre*, in land measure, and its fourth part, called a *rood*. There is no such linear measure as an acre or a rood. It would be absurd to speak of an acre long or a rood long.

The usual units of square measure are given by the following

TABLE OF SQUARE MEASURE.

<i>sq. inches.</i>	<i>sq. feet.</i>	<i>sq. yd.</i>	<i>sq. rd.</i>	<i>sq. ch's.</i>	<i>S. M.</i>
144 =	1				
1296 =	9 =	1			
39204 =	272 $\frac{1}{4}$ =	30 $\frac{1}{4}$ =	1		
627264 =	4356 =	484 =	16 =	1	<i>Acres.</i>
6272640 =	43560 =	4840 =	160 =	10 =	1
4014489600 =	27878400 =	3097600 =	102400 =	6400 =	640 = 1

§ 85. In measuring land, Gunter's Chain is most commonly employed. Its length is 4 rods, or 66 feet, and is divided into 100 links, so that we have as follows :

$7\frac{92}{100}$	inches, or 7.92 inches = ...	1 link.
100	links=792 inches=66 feet=22 yards=4 rods=	1 chain.
80	chains = ...	1 mile.
10000	sq. links = 16 sq. rods = ...	1 sq. chain.
100000	sq. links = 10 sq. chains = 160 sq. rods = .	1 acre.

The unit of measure for land is the *acre*. It cannot be given in an exact square, since it is made to consist of 160 square rods, and 160 is not a square number, which proves that the side of a square containing 160 sq. rods cannot be accurately expressed in rods; and if not in rods, then not in any multiple or submultiple of rods.

One-fourth of an acre is called a *rood*, and the square rod in land measure is frequently called a *perch*.

If links be multiplied by links, the product will be square links, which may be reduced to acres by dividing by 100000, or by simply pointing off five decimal places. If chains be multiplied by chains, the product will be square chains, which may be reduced to acres by dividing by 10, or pointing off one decimal figure. If chains are multiplied by links, the product will be converted into acres by pointing off three decimals.

We will also add, that to convert square feet into acres, we must divide by 43560. To convert square yards into acres, we must divide by 4840; and to convert square rods into acres, we must divide by 160.

PROBLEM I.

To find the area of a square or rectangular piece of land.
The obvious rule in this case will be as follows :

Take the product of two adjacent sides for the area. [Geom., B. III., T. XXI., Scholium.]

EXAMPLES.

1. How many acres in a piece of ground in the form of a square, each of whose sides is 13 chains, 25 links, or 13.25 chains?

$$13.25 \times 13.25 = 175.5625 \text{ sq. chains.}$$

Or, since $13\cdot25$ chains = 1325 links, we have

$$1325 \times 1325 = 1755625 \text{ square links,}$$

$$\text{and } 17\cdot55625 = \text{the number of acres.}$$

Converting this decimal of an acre into roods and perches, we have

$$\begin{array}{r} 0\cdot55625 \\ \underline{4} \\ \text{roods} = 2\cdot22500 \\ \underline{40} \\ \text{perches} = 9\cdot00000 \end{array} \quad \text{Ans. } 17A. 2R. 9P.$$

2. How many acres in a rectangular piece of land whose adjacent sides are $30\cdot25$ chains and $25\cdot21$ chains?

$$2521 \times 3025 = 7626025 \text{ sq. links} = 76A. 1R. 1\frac{64}{100}P.$$

3. How many rods in a rectangular piece of ground 282 feet wide and 325 feet long?

$$\text{Ans. } 336\cdot639 \text{ sq. rods.}$$

PROBLEM II.

To find the area of a parallelogram.

FIRST RULE.

When the altitude is given, multiply it by the base, for the area. [Geom., B. III., T. XXII.]

If we multiply AD by the sine of the angle at A, we shall obtain DE. (Case I., Right Triangles.) That is,

$$DE = AD \times \sin. A,$$

$$\text{and area } ABCD = DE \times AB = AD \times AB \times \sin. A. \quad (1.)$$

Using logarithms,

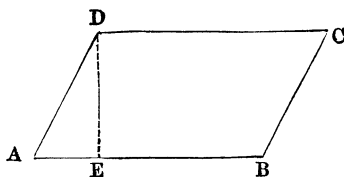
$$\log. \text{ area} = \log. AD + \log. AB + \log. \sin. A - 10. \quad (2.)$$

We subtract 10 to correct for the 10 added to log. sin. of A.

Hence we have this

SECOND RULE.

Multiply the product of two adjacent sides by the natural sine of the included angle.



OR BY LOGARITHMS.

To the sum of the logarithms of two adjacent sides add the logarithmic sine of the included angle, and subtract 10, and the result will be the logarithm of the area.

EXAMPLES.

1. How many acres in a field in the form of a parallelogram, having a base of 13 chains 14 links, and an altitude of 10 chains 37 links? $1314 \times 1037 = 1362618$ sq. links = 13A. 2R. 20P.

2. How many acres in a field in the form of a parallelogram, the adjacent sides being 25·17 chains and 30·25 chains, and having $70^\circ 30'$ for the included angle?

BY LOGARITHMS.

$$\log. 25\cdot17 = 1\cdot400883$$

$$\log. 30\cdot25 = 1\cdot480725$$

$$\sin. 70^\circ 30' = 9\cdot974347$$

$$\log. \text{area} = 2\cdot855955$$

Hence, area = 717·72 sq. chains, which is

Ans. 71A. 3R. $3\frac{1}{2}$ P.

3. A field, in the form of a parallelogram, has 10·21 chains, 12·12 chains for the adjacent sides, and $30^\circ 45'$ for the included angle. How many acres does it contain?

Ans. 63·27 sq. chains.

PROBLEM III.

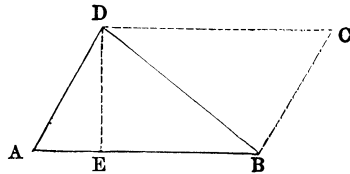
To find the area of a triangle.

FIRST RULE.

When the altitude is given, multiply its half by the base.
[Geom., B. III. T. XXIII.]

Or multiply the altitude by half the base, or take half the product of the altitude and base.

Since the triangle ABD is obviously just one-half of the par-



allelogram ABCD, whose measure is given by equation (1) of Prob. II., we obviously have

$$\text{area ABD} = \frac{DE \times AB}{2} = \frac{AD \times AB \times \sin. A}{2}. \quad (1.)$$

Or, using logarithms,

$$\log. \text{area} = \log. AD + \log. AB + \log. \sin. A - 10.301030. \quad (2.)$$

The log. of 2, the denominator of (1), being 0.301030, is to be subtracted in addition to the 10.

Hence, we have this

SECOND RULE.

Multiply the product of any two adjacent sides by the sine of the included angle, and one-half this last product will be the area.

BY LOGARITHMS.

To the sum of the logarithms of any two adjacent sides add the logarithmic sine of the included angle, and subtract 10.301030; and the remainder will be the logarithm of the area.

We have also given [Geom., B. VIII., P. IV.] a demonstration to this

THIRD RULE.

Take half the sum of the three sides, and from this half sum subtract each side separately; then take the square root of the continued product of the half sum and the three remainders, and it will be the area.

BY LOGARITHMS.

Find the half sum of the three sides, and the three remainders as before; take half the sum of their logarithms, and it will be the logarithm of the area.

EXAMPLES.

1. How many square feet in a triangle whose base is 110.5 feet and altitude 97.25 feet?

Ans. $\frac{1}{2}$ of $110.5 \times 97.25 = 5373.0625$ sq. feet.

2. If an angle of a triangle is $33^\circ 33'$, included by sides whose

lengths are 45·25 feet and 37·5 feet respectively, what will be its area in square rods?

$$\log. 45\cdot25 = 1\cdot655619$$

$$\log. 37\cdot5 = 1\cdot574031$$

$$\sin. 33^\circ 33' = 9\cdot742462$$

$$12\cdot972112$$

$$10\cdot301030$$

$$2\cdot671082 = \log. \text{ of } sq. \text{ feet.}$$

$$\text{subtract } 2\cdot434968 = \log. \text{ of } 272\frac{1}{2}, sq. ft. \text{ in a } sq. rd.$$

$$0\cdot236114 = \log. \text{ of } sq. \text{ rods.}$$

$$Ans. 1\cdot7223 \text{ } sq. \text{ rods.}$$

3. How many acres in a triangular field whose sides are 70, 110, and 120 chains, respectively?

$$\frac{1}{2} \text{ of } (70 + 110 + 120) = 150, \log. = 2\cdot176091$$

$$150 - 70 = 80, \log. = 1\cdot903090$$

$$150 - 110 = 40, \log. = 1\cdot602060$$

$$150 - 120 = 30, \log. = 1\cdot477121$$

$$2) \overline{7\cdot158362}$$

$$\log. \text{ area} = 3\cdot579181$$

$$Ans. 3794\cdot7 \text{ } sq. \text{ chains} = 379\cdot47 \text{ acres.}$$

4. How many *sq.* feet in a triangle, each of whose sides is denoted by a ?

$$Ans. \sqrt{\frac{3}{2} a \times \frac{1}{2} a \times \frac{1}{2} a \times \frac{1}{2} a} = \frac{a^2}{4} \sqrt{3}.$$

5. Referring to the last example, how many *sq.* feet in an equilateral triangle whose side is 10 feet?

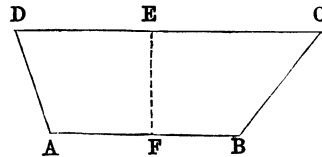
$$Ans. 43\cdot301 \text{ } sq. \text{ feet.}$$

PROBLEM IV

To find the area of a trapezoid.

RULE.

Multiply half the sum of the parallel sides by the perpendicular distance between them. [Geom., B. III., T. XXIII., Cor. II.]



EXAMPLES.

1. What is the area of a trapezoid, whose parallel sides are 20 and 30 chains, and which are perpendicularly distant 12·5 chains?

$$\begin{aligned}\frac{1}{2} \text{ of } (20 + 30) &= 25; \\ 25 \times 12\cdot5 &= 312\cdot5 \text{ sq. chains, or} \\ &31\frac{1}{4} \text{ acres.}\end{aligned}$$

2. If the parallel sides of a trapezoid, whose lengths are 25 and 35 feet, are at the distance of 12 feet from each other, what will be its area?

Ans. 360 sq. feet.

3. What is the area of a trapezoid, whose parallel sides are 3 and 4·90 chains, and the perpendicular distance between them 1·66 chains?

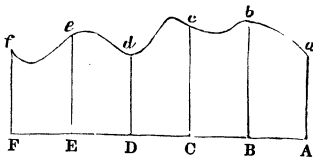
Ans. 6·557 sq. chains.

4. How many square feet in a trapezoidal table whose parallel sides are 5·5 feet and 4·25 feet, the width being 3·5 feet?

Ans. 17·0625 sq. feet.

SCHOLIUM.—In land surveying, we frequently have occasion to measure long irregular portions of a field, which are bounded on one side by a straight line.

At equal intervals along the straight line AF, as at B, C, D, and E, measure the perpendiculars Bb, Cc, Dd, &c.; also measure the extreme perpendiculars Aa and Ff. When the equal intervals AB, BC, CD, &c., are short, the figures ABba, BCcb, CDdc, &c., may, without much error, be regarded as trapezoids; hence, calling *s* one of the equal distances, AB, BC, CD, &c., we have



$$\text{area ABba} = \frac{Aa + Bb}{2} \times s,$$

$$\text{area BCcb} = \frac{Bb + Cc}{2} \times s,$$

$$\text{area CDdc} = \frac{Cc + Dd}{2} \times s,$$

$$\text{area DEed} = \frac{Dd + Ee}{2} \times s,$$

$$\text{area EFfe} = \frac{Ee + Ff}{2} \times s.$$

The total area is $(\frac{1}{2} Aa + Bb + Cc + Dd + Ee + \frac{1}{2} Ff) \times s$.

Hence this

RULE.

To half the sum of the extreme breadths add all the intermediate breadths ; multiply this sum by one of the equal distances, and the product will be the area.

As an example, suppose seven breadths of an irregular field, taken at equal intervals, to be 32 links, 47 links, 63 links, 13 links, 17 links, 5 links, and 2 links ; if one of the equal intervals is 10 links, what will be the area ?

$$\frac{32 + 2}{2} + 47 + 63 + 13 + 17 + 5 = 162,$$

and $162 \times 10 = 1620$ for the number of square links in the area.

PROBLEM V.

To find the area of any irregular polygonal figure.

RULE.

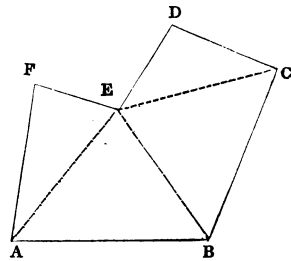
Draw a sufficient number of diagonals to divide the whole figure into triangles. Then compute each triangle separately, and take their sum.

As an illustration, suppose ABCDEF to be the polygon. Draw the diagonals EA, EB, EC, and suppose by actual measurement we find AB = 12·5 ; BC = 11·25 ; CD = 7 ; DE = 6·75 ; EF = 5·75 ; FA = 10·5 ; EA = 10·75 ; EB = 10 ; EC = 9·75.

Now computing each triangle by the *Third Rule* under Problem III., we shall find

$$\begin{aligned} \text{Area EFA} &= 29\cdot3795 \\ \text{" EAB} &= 51\cdot6642 \\ \text{" EBC} &= 45\cdot6431 \\ \text{" ECD} &= 23\cdot6247 \end{aligned}$$

$$\text{area ABCDEF} = 150\cdot3115$$

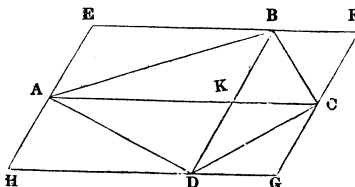


PROBLEM VI.

To find the area of a quadrilateral in terms of its diagonals and their included angle.

Let ABCD be the quadrilateral figure, of which the diagonals AC, BD, and their angle K of intersection are given.

Through A, B, C, D, draw lines parallel to the diagonals, which will together form the parallelogram EFGH, whose angles are equal to the angles at K; and whose area is obviously double that of the quadrilateral ABCD.



But, Prob. II., $EFGH = HE \times HG \times \sin. H = BD \times AC \times \sin. K$; and we have $ABCD = \frac{1}{2} BD \times AC \times \sin. K$.

Hence we have this

RULE.

Multiply the product of the two diagonals by the sine of their included angle, and one-half this last product will be the area.

BY LOGARITHMS.

To the sum of the logarithms of the two diagonals add the logarithmic sine of their included angle, and subtract 10·301030; and the remainder will give the logarithm of the area.

NOTE.—This rule will agree with the second rule of Prob. III., if the word *diagonals* is changed into *sides*.

EXAMPLES.

1. The diagonals of a quadrilateral are 37·5 and 46·25, and they make with each other an angle of $47^\circ 47'$. What is the area?

$$\log. 37\cdot5 = 1\cdot574031$$

$$\log. 46\cdot25 = 1\cdot665112$$

$$\sin. 47^\circ 47' = 9\cdot869589$$

$$\hline 13\cdot108732$$

$$10\cdot301030$$

$$\hline \log. \text{ area} = 2\cdot807702$$

$$\text{and area} = 642\cdot247.$$

2. The diagonals of a quadrilateral are 90 and 100, and make an angle of 75° with each other. What is the area?

Ans. 4346.68.

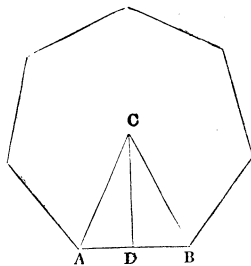
PROBLEM VII.

To find the area of any regular polygon.

FIRST RULE.

Take one-half the product of the perimeter and perpendicular drawn from its centre to one of its sides. [Geom., B. IV., T. V.]

When the perpendicular is not given, it may be found as follows: the angle ACB will be found by dividing 360° by the number of sides in the polygon; and the angle BCD, which is one-half of ACB, can be found by dividing 180° by the number of sides. Hence if we denote the number of sides by n , we shall have $\frac{180^\circ}{n}$, for the angle BCD.



Now BD, which is one-half of a side of the polygon, being multiplied by $\cotan. BCD$, will give CD. If, then, s denote a side and p the perpendicular, we shall have

$$p = \frac{1}{2} \text{ of } s \times \cotan. \frac{180^\circ}{n}. \quad (1.)$$

And, the perimeter being $n \times s$, we shall obtain

$$\text{area} = \frac{1}{2} \text{ of } p \times n \times s = \frac{1}{4} \text{ of } s^2 \times n \times \cotan. \frac{180^\circ}{n}. \quad (2.)$$

If the side of the polygon is taken as a unit, then $s = 1$, and

$$\text{area} = \frac{1}{4} \text{ of } n \times \cotan. \frac{180^\circ}{n}. \quad (3.)$$

If, in formula (3), we take successively $n = 3, 4, 5, 6, 7$, &c., we shall obtain the areas of regular polygons, whose sides are unity, as given in the following

TABLE.

<i>Names.</i>	<i>Sides.</i>	<i>Areas.</i>	<i>Logarithms.</i>
Triangle	3.....	0·4330127.....	1·6365007
Square	4.....	1·0000000.....	0·0000000
Pentagon	5.....	1·7204774.....	0·2356490
Hexagon	6.....	2·5980762.....	0·4146519
Heptagon	7.....	3·6339124.....	0·5603744
Octagon	8.....	4·8284271.....	0·6838057
Nonagon.....	9.....	6·1818242.....	0·7911166
Decagon.....	10.....	7·6942088.....	0·8861640
Undecagon.....	11.....	9·3656399.....	0·9715375
Dodecagon.....	12.....	11·1961524.....	1·0490687

Since the areas of similar figures are to each other as the squares of their homologous sides, we may find the area of any regular polygon whose number of sides is not greater than 12 by this

SECOND RULE.

Multiply the square of the length of a side of the polygon by the area of a similar polygon whose side is unity, as found in the preceding table.

BY LOGARITHMS.

To twice the logarithm of a side of the given polygon add the logarithm of the corresponding tabular number.

EXAMPLES.

1. What is the area of a regular dodecagon whose side is 7·75?

$$\begin{aligned}
 \log. 7\cdot75 &= 0\cdot889302 \\
 \log. 7\cdot75 &= 0\cdot889302 \\
 \text{tabular log.} &= 1\cdot049069 \\
 \log. \text{ area} &= 2\cdot827673 \\
 \text{Area} &= 672\cdot469.
 \end{aligned}$$

2. What is the area of an octagonal room, each side of which is 10·5 feet? *Ans.* 532·334 sq. feet.

3. What is the number of square inches in a pentagon, each of whose sides is 10 inches? *Ans.* 172·048 sq. inches.

4. What is the area of an equilateral triangle, having 100 for a side? *Ans.* 4330·127.

PROBLEM VIII.

To find the circumference of a circle when its diameter is known, and conversely.

If the diameter of a circle is a unit, its circumference will be 3·141592653589793, &c. (§ 5). Denote this ratio by π , and put

R = radius,

D = diameter,

C = circumference.

Now since circumferences are to each other as their diameters, and consequently as their radii, we shall have

$$C = D \times \pi = D \times 3\cdot14159265, \quad . \quad . \quad (1.)$$

$$C = R \times 2\pi = R \times 6\cdot28318531, \quad . \quad . \quad (2.)$$

$$D = C \times \frac{1}{\pi} = C \times 0\cdot31830989, \quad . \quad . \quad (3.)$$

$$R = C \times \frac{1}{2\pi} = C \times 0\cdot15915494. \quad . \quad . \quad (4.)$$

Using logarithms, we have

$$\log. C = 0\cdot4971499 + \log. D, \quad . \quad . \quad (5.)$$

$$\log. C = 0\cdot7981799 + \log. R, \quad . \quad . \quad (6.)$$

$$\log. D = \bar{1}\cdot5028501 + \log. C, \quad . \quad . \quad (7.)$$

$$\log. R = \bar{1}\cdot2018201 + \log. C. \quad . \quad . \quad (8.)$$

EXAMPLES.

1. What is the circumference of a circle whose diameter is 12·5 feet? *Ans.* $3\cdot1416 \times 12\cdot5 = 39\cdot27$ feet, nearly.

2. If the radius of the earth is 3956 miles, what is its circumference? *Ans.* 24856·281 miles, nearly.

3. If the circumference of a circle is 100 feet, what is its diameter? *Ans.* $0\cdot31830989 \times 100 = 31\cdot83$ feet, nearly.

4. If the circumference of a wheel is 15·5 feet, what is its radius? *Ans.* 2·4669 feet.

5. What is the circumference of a circle whose diameter is 100 feet? *Ans.* 314·159 feet.

PROBLEM IX.

To find the area of a circle when its diameter is known, and conversely. Also when the circumference is known.

Denoting the area by A and continuing the notation as under Prob. VIII., we have [Geom., B. IV., T. XIII.]

$$A = \frac{1}{4} \text{ of } D \times C.$$

Substituting the value of C as already given, we have

$$A = D^2 \times \frac{\pi}{4} = D^2 \times 0.78539816, \quad . \quad . \quad (1.)$$

$$A = R^2 \times \pi = R^2 \times 3.14159265, \quad . \quad . \quad (2.)$$

$$A = C^2 \times \frac{1}{4\pi} = C^2 \times 0.07957747, \quad . \quad . \quad (3.)$$

$$D = \sqrt{A} \times 2\sqrt{\frac{1}{\pi}} = \sqrt{A} \times 1.12837917, \quad . \quad (4.)$$

$$R = \sqrt{A} \times \sqrt{\frac{1}{\pi}} = \sqrt{A} \times 0.56418958, \quad . \quad (5.)$$

$$C = \sqrt{A} \times 2\sqrt{\pi} = \sqrt{A} \times 3.54490770. \quad . \quad (6.)$$

Using logarithms, we have

$$\log. A = \bar{1}.8950899 + 2 \times \log. D, \quad . \quad (7.)$$

$$\log. A = 0.4971499 + 2 \times \log. R, \quad . \quad (8.)$$

$$\log. A = \bar{2}.9007901 + 2 \times \log. C, \quad . \quad (9.)$$

$$\log. D = 0.0524551 + \frac{1}{2} \times \log. A, \quad . \quad (10.)$$

$$\log. R = \bar{1}.7514251 + \frac{1}{2} \times \log. A, \quad . \quad (11.)$$

$$\log. C = 0.5496049 + \frac{1}{2} \times \log. A. \quad . \quad (12.)$$

EXAMPLES.

1. What is the area of a circle, whose diameter is 125 feet?

By equation (7), we have

$$\begin{aligned} & \bar{1}.895090 \\ 2 \times \log. 125 &= 4.193820 \\ \hline \log. A &= 4.088910 \\ A &= 12271.8. \end{aligned}$$

If more decimals are required, they may be obtained by using equation (1), which gives $0.78539816 \times 125 \times 125 = 12271.84625$.

2. What is the number of square inches in a circular stand-top, whose radius is 15 inches?

Ans. $3.1416 \times 15 \times 15 = 706.86$ sq. inches.

3. What is the area of a circular fish-pond, whose circumference is 25 rods?

Ans. $0.07957747 \times 25 \times 25 = 49.734$ sq. rods.

4. What is the number of rods in the diameter, the radius, and the circumference of a circle, which shall contain just 1 acre?

Using equations (10), (11), and (12), we have

$$\begin{array}{r r r} 0.052455 & \bar{1}.751425 & 0.549605 \\ \frac{1}{2} \text{ of log. } 160 = 1.102060 & \dots 1.102060 & \dots 1.102060 \\ \hline 1.154515 & 0.853485 & 1.651665 \\ D = 14.273. & R = 7.1365. & C = 44.84. \end{array}$$

NOTE.—This last example would have required far more labor had we used equations (4), (5), and (6), which are independent of logarithms.

5. How many chains must the diameter of a circular farm be which shall contain just 100 acres? *Ans.* 35.68 chains.

PROBLEM X.

To determine formulas for the area and for the arc of a sector when the angle at the centre is known, and conversely.

Denote the number of degrees which measures the angle at the centre by n , the length of the corresponding arc by c , and the area of corresponding sector by a . Then, since arcs of the same circle are to each other as their corresponding angles at the centre; also, since sectors are to each other in the same ratio, we shall have

$$360 : n :: C : c,$$

$$360 : n :: A : a.$$

And

$$c = \frac{n}{360} \times C, \quad . \quad . \quad . \quad (1.)$$

$$a = \frac{n}{360} \times A. \quad . \quad . \quad . \quad (2.)$$

EXAMPLES.

1. What is the length of an arc of 15° of the earth, its entire circumference being 24856.28 miles?

Ans. $\frac{15}{360}$ of 24856.28 = 1035.68 miles, nearly.

2. What is the area of a sector of 10 degrees of a circle, whose diameter is 10 feet?

If, in equation (2), we substitute the value of A as given by equation (1) of Prob. IX., it will become

$$a = \frac{n}{360} \times \frac{\pi}{4} \times D^2 = \frac{10}{360} \times \frac{\pi}{4} \times 100 = 2.18 \text{ sq. feet, nearly.}$$

3. What is the length of an arc of 1° of the moon, its diameter being 2160 miles?

In equation (1), substitute the value of C as given by (1) of Prob. VIII., and we have

$$c = \frac{n}{360} \times \pi \times D = \frac{1}{360} \times \pi \times 2160 = 18.849 \text{ miles, nearly.}$$

4. An arc on the earth of 500 miles corresponds to how great an angle at the centre?

Equation (1) gives immediately [see equation (1), Prob. IX.]

$$n = \frac{c}{C} \times 360^\circ = \frac{500}{24856.28} \times 360^\circ = 7.242 = 7^\circ 14' 31''.$$

5. A sector whose area is 10 square feet corresponds to how great an angle at the centre, provided the diameter of the circle is 10 feet?

Equation (1) gives immediately [see equation (1), Prob. IX.]

$$n = \frac{a}{A} \times 360^\circ = \frac{a}{D^2} \times \frac{4}{\pi} \times 360^\circ = \frac{10}{100} \times \frac{4}{\pi} \times 360^\circ = 45.8366 = 45^\circ 50' 12''.$$

PROBLEM XI.

To find the area of a segment of a circle.

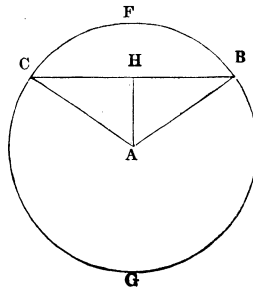
If n denote the number of degrees in the angle BAC , we shall have [Prob. X.]

$$\text{area of sector ABFC} = \frac{n}{360} \times R^2 \times \pi,$$

since $A = R^2 \times \pi$, by equation (5), Prob. IX.,

$$\text{area of triangle ABC} = \frac{1}{2} R^2 \times \sin. n.$$

[Prob. III., Second Rule.]



Hence the area of the segment BHCF, being the difference between the sector and triangle, is

$$\left(\frac{n}{360} \times \pi - \frac{1}{2} \sin. n\right) \times R^2. \quad . \quad . \quad (1.)$$

And the area of the segment BHCG, which is greater than a semicircle, is the sum of the sector CBG and triangle ABC. Hence, to determine the area of a segment, we have this

RULE.

Find the area of a sector which has the same arc as the segment; also, the area of the triangle formed by the chord of the segment and the radii of the sector.

Then take the sum of these areas when the segment exceeds the semicircle, and their difference when it is less.

EXAMPLES.

1. In a circle whose radius is 10 feet, find the area of a segment whose arc corresponds with an angle of 10° at the centre.

$$\text{Area of sector} = \frac{n}{360} \times \pi \times R^2 = \frac{10}{360} \times 3.14159 \times 10^2 = 8.727.$$

$$\text{" " triangle} = \frac{1}{2} \sin. 10^\circ \times 10^2 = \frac{1}{2} \text{ of } 0.17365 \times 10^2 = 8.683.$$

$$\text{The difference} = \text{area of segment} = 0.044 \text{ of a sq. foot.}$$

2. In a circle whose radius is 15 rods, it is required to find the area of a segment whose arc corresponds to a central angle of 200° .

$$\text{Area of sector} = 392.699$$

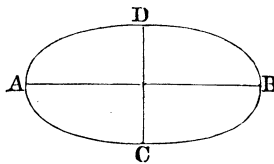
$$\text{" of triangle} = 38.477$$

$$\text{" of segment} = 431.176 \text{ sq. rods.}$$

PROBLEM XII.

To find the area of an ellipse.

A line drawn through the *centre* of an ellipse is called its *diameter*. The longest diameter is called the *transverse* diameter; the shortest is called the *conjugate* diameter. Thus AB is the transverse diameter, and CD is the conjugate diameter. These diameters mutually bisect each other at right angles.



The area of an ellipse may be found by this

RULE.

Multiply the product of the transverse and conjugate diameters by $0.7854 = \frac{1}{4} \pi$.

A demonstration of this rule cannot be given by the elementary principles of Geometry.

EXAMPLES.

1. How many square feet in the surface of an elliptical pond, whose transverse diameter is 100 feet, and conjugate diameter 60 feet? *Ans.* $100 \times 60 \times 0.7854 = 4712.4$ square feet.

2. How many square inches is an elliptical table, whose transverse diameter is 5 feet 3 inches, and conjugate diameter 3 feet 6 inches? And how many square feet?

Ans. $\left\{ \begin{array}{l} 2078.1684 \text{ square inches.} \\ 14.4317 \text{ square feet.} \end{array} \right.$

CHAPTER IX.

MENSURATION OF SOLIDS,

Which includes the measure of their surfaces as well as that of their volumes.

§ 86. THE common unit for measuring solids is a cube; as, a cubic inch, a cubic foot, a cubic yard, &c. The values of the usual units of solid measure are given in the following

TABLE.

1728	cubic inches	= 1 cubic foot.
27	cubic feet	= 1 cubic yard.
$4492\frac{1}{8}$	cubic feet	= 1 cubic rod.
128	cubic feet	= 1 cord of wood.
231	cubic inches	= 1 gallon (liquid measure).
$268\frac{4}{5}$	cubic inches	= 1 gallon (dry measure).
$2150\frac{2}{5}$	cubic inches	= 1 bushel.

§ 87. By an act of Parliament of Great Britain, which took effect on the 1st of January, 1826, the Imperial gallon of 277·274 cubic inches was adopted as the only gallon. This value was found to be the measure of 10 pounds, avoirdupois, of distilled water. Estimating 8 gallons to the bushel, we have for the Imperial bushel 2218·192 cubic inches.

New York as well as the other States of the Union still continue to use the old English gallons as given in the foregoing table.

PROBLEM I.

To find the surface of a right prism, or of a right cylinder.

RULE.

Multiply the perimeter of the base by the altitude, for the convex surface. To which add the areas of the two bases if the entire surface is required. [See Geom., B. VI., T. I. Also B. VII., T. I.]

EXAMPLES.

1. What is the entire surface of an octagonal prism, whose height is 25 feet, and each side of its base being 1 foot 3 inches?

$$1\cdot25 \times 8 \times 25 = \text{convex surface } 250 \text{ sq. feet.}$$

area of bases 7·544

$$257\cdot544 \text{ sq. feet.}$$

2. What is the convex surface of a cylinder, whose length is 40 feet, and its diameter 2 feet? *Ans.* 251·328 sq. feet.

PROBLEM II.

To find the volume of a right prism, or of a right cylinder.

RULE.

Multiply the area of the base by the altitude. [See Geom., B. VI., T. XII., and B. VII., T. II.]

EXAMPLES.

1. What is the volume of a triangular prism, whose height is 20 feet, and each side of whose base is 2 feet?

Ans. 34·641 cubic feet.

2. How many Imperial gallons in a cylindrical oil-can, the diameter being 20 inches, and depth 40 inches?

Denoting the diameter, in inches, by d , and height by h , and putting g for the number of inches in a gallon, we have

$$\text{Number of gallons} = \frac{\pi}{4g} \times d^2 \times h. \quad (1.)$$

For Imperial gallons, $g = 277.274$; for United States, or old English gallons, $g = 231$. Hence

$$\text{Imperial gallons} = 0.0028326 \times d^2 \times h, \quad (2.)$$

$$\text{United States gallons} = 0.0034 \times d^2 \times h. \quad (3.)$$

USING LOGARITHMS.

$$\log. [\text{Imp. galls.}] = \bar{3}.4521808 + 2 \times \log. d + \log. h, \quad (4.)$$

$$\log. [\text{U. S. galls.}] = \bar{3}.5314779 + 2 \times \log. d + \log. h. \quad (5.)$$

Now, by equation (2), we find

$$\text{Ans. } 0.0028326 \times 20^2 \times 40 = 45.32 \text{ Imp. gallons.}$$

3. How many cubic feet in a cylindric log 14 feet long, and 14 inches in diameter? *Ans.* 14.966 cubic feet.

4. How many cubic inches in a cylindric measure of $18\frac{1}{2}$ inches diameter, and 8 inches deep?

$$\text{Ans. } 2150.425 \text{ cubic inches.}$$

NOTE.—The dimensions of the measure as given in the last example are those of the old English bushel, usually called the Winchester bushel; so named from the place where it was deposited for safe keeping.

5. How many United States gallons of oil in a can of 18 inches diameter, the oil being only 13 inches deep?

$$\text{Ans. } 14.32 \text{ U. S. gallons.}$$

PROBLEM III.

To find the surface of a regular pyramid, or of a conc.

RULE.

Multiply the perimeter of the base by half the slant height. Add the surface of base when the entire surface is required.

[See Geom., B. VI., T. XV. Also B. VII., T. III.]

EXAMPLES.

1. What is the convex surface of a cone whose slant height is 20 feet, and the diameter of the base 6 inches?

Ans. 15·708 sq. feet.

2. What is the convex surface of an octagonal pyramid, its base being 10 inches on each side, and the slant height being 35 feet?

Ans. 116 $\frac{2}{3}$ sq. feet.

PROBLEM IV.

To find the volume of a regular pyramid, or of a cone.

RULE.

Multiply the area of the base by one-third of the altitude.
[See Geom., B. VI., T. XVIII., and B. VII., T. V.]

EXAMPLES.

1. How many cubic feet in a conical stick of timber 40 feet long, and 20 inches diameter at the base?

Ans. 29·0888 cubic feet.

2. How many cubic yards in a conical stack of hay, which is 18 feet in diameter and 27 feet high?

Ans. 84·823 cubic yards.

3. How many cubic feet in a regular octagonal pyramid of marble, whose height is 30 feet, and a side of the base 2 feet?

Ans. 193·136 cubic feet.

PROBLEM V.

To find the surface of a frustum of a regular pyramid, or of a frustum of a cone.

RULE.

Multiply half the sum of the perimeters of the two bases by the slant height. When the entire surface is required, add the areas of the two bases.

[See Geom., B. VI., T. XV., Cor. II. Also B. VII., T. IV.]

EXAMPLES.

1. What is the entire surface of a frustum of a cone, the slant height of which is 15 feet, and the diameter of one base 2 feet 6 inches, and that of the other 1 foot 3 inches?

Ans. 94·4934 sq. feet.

2. What is the convex surface of a frustum of a regular hexagonal pyramid, the slant height being 12 feet, each side of the larger base being 4 feet, and each side of the smaller base 3 feet?

Ans. 252 sq. feet.

PROBLEM VI.

To find the volume of a frustum of a regular pyramid, or of a cone.

If we denote the volume of the frustum by V , we shall have (Geom. B. VI., T. XIX.), (See also B. VII., T. V., C. II.)

$$V = (A + a + \sqrt{A \times a}) \times \frac{1}{3} h = A \times \frac{1}{3} h + a \times \frac{1}{3} h + \sqrt{A \times a} \times \frac{1}{3} h.$$

Hence the following

RULE.

Add together the areas of the two bases, and a mean proportional between them, and multiply this sum by one third of the altitude.

In the case of the cone, if D and d denote the diameters of the bases, we shall have

$$A = D^2 \times \frac{\pi}{4}; \quad a = d^2 \times \frac{\pi}{4}, \text{ and } A^{\frac{1}{2}} a^{\frac{1}{2}} = Dd \times \frac{\pi}{4}.$$

And
$$V = (D^2 + d^2 + Dd) \times h \times 0.26179939. \quad (1.)$$

If D , d , and h are estimated in inches, we shall have the number of United States gallons, which is the same as the number of old English gallons, in this volume, by dividing by 231. If we divide by 277·274, we shall obtain the number of Imperial gallons. Thus,

$$\text{U.S. gallons} = (D^2 + d^2 + Dd) \times h \times 0.0011333, \quad (2.)$$

$$\text{Imp. gallons} = (D^2 + d^2 + Dd) \times h \times 0.0009442. \quad (3.)$$

Using logarithms, we have

$$\log. [\text{U. S. galls.}] = \bar{3}.0543566 + \log. (D^2 + d^2 + Dd) + \log. h, \quad (4.)$$

$$\log. [\text{Imp. galls.}] = \bar{4}.9750595 + \log. (D^2 + d^2 + Dd) + \log. h. \quad (5.)$$

EXAMPLES.

1. A block of marble is in the form of a frustum of a square pyramid: the larger base is 4 feet each way, and the smaller base is 3 feet on a side; the height is 6 feet. How many cubic feet does it contain?

$$A = 4^2 = 16; \quad a = 3^2 = 9; \quad \text{and } A^{\frac{1}{2}} a^{\frac{1}{2}} = 12.$$

$$V = (A + a + A^{\frac{1}{2}} a^{\frac{1}{2}}) \times \frac{1}{3} h = (16 + 9 + 12) \times 2 = 74 \text{ cubic ft.}$$

2. A cistern in form of a frustum of a cone is 9 feet deep, having for diameters 8 feet and 10 feet. How many U. S. gallons of 231 cubic inches will it contain?

Using equation (4), we have

$$D^2 + d^2 + Dd = 35136; \quad h = 108.$$

$$\overline{3.054357}$$

$$\log. 35136 = 4.545752$$

$$\log. 108 = 2.033424$$

$$\log. [\text{U. S. galls.}] = 3.633533 \qquad \text{Ans. } 4300.6 \text{ U. S. galls.}$$

3. How many Imperial gallons in a cistern in the form of a frustum of a cone, the bottom diameter being 12 feet and top diameter being 10 feet, the depth also 10 feet?

Using equation (5), we have

$$D^2 + d^2 + Dd = 52416; \quad h = 120.$$

$$\overline{4.975060}$$

$$\log. 52416 = 4.719464$$

$$\log. 120 = 2.079181$$

$$\log. [\text{Imp. galls.}] = 3.773705 \qquad \text{Ans. } 5938.9 \text{ Imp. galls.}$$

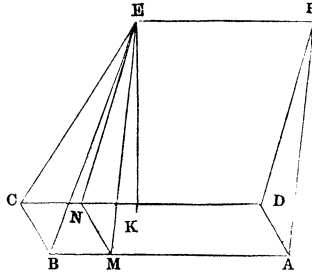
4. How many cubic feet in a block of marble in the form of the frustum of a regular hexagonal pyramid, each side of the larger base being 4 feet, and each side of the smaller base being 3 feet, and the altitude being 9 feet?

$$\text{Ans. } 288.386 \text{ cubic feet.}$$

PROBLEM VII.

To find the volume of a wedge.

Definition.—A *wedge* is a solid bounded by five planes; a rectangle ABCD, called the base of the wedge; two trapezoids ABEF, DCEF, called the sides of the wedge, and which intersect in the line EF, parallel to the base, which is called the edge of the wedge; and two triangles ADF, BCE, which are called the ends of the wedge. If through E we pass the plane EMN parallel to the end FAD, the portion EMBCN thus cut off is a pyramid, and the remaining wedge having AMND for its base is evidently a triangular prism, and has for its measure one-half of a parallelipedon, having the same base and same altitude.



Hence if $L = AB$, the length of base, $l = EF$, the length of the edge $b = AD$, the breadth of base, and $h = EK$, the altitude, then we shall have

$$\begin{aligned} \text{Volume of the wedge whose base is AMND} \\ = \frac{1}{2} \text{ of } AM \times AD \times EK = \frac{1}{2} lbh. \end{aligned} \quad (1.)$$

$$\begin{aligned} \text{Volume of pyramid EMBCN} &= \frac{1}{3} \text{ of } MB \times BC \times EK \\ &= \frac{1}{3} (L - l) bh = \frac{1}{3} Lbh - \frac{1}{3} lbh. \end{aligned} \quad (2.)$$

Taking the sum, we have for the volume of the entire wedge

$$\frac{1}{3} Lbh + \frac{1}{6} lbh = \frac{1}{6} (2L + l) bh. \quad (3.)$$

If the length of the wedge is less than the length of the edge, the volume sought will be the difference between the prism and pyramid. And in this case the volume of the pyramid will be $\frac{1}{3} (l - L) bh = \frac{1}{3} lbh - \frac{1}{3} Lbh$, which subtracted from $\frac{1}{2} lbh$, the volume of the prism, we have

$$\frac{1}{6} lbh + \frac{1}{3} Lbh = \frac{1}{6} (2L + l) bh, \quad (4.)$$

which is precisely the same as expression (3).

Hence the following

RULE.

To twice the length of the base add the edge, multiply the sum by the breadth of the base, and that product by $\frac{1}{6}$ of the height of the wedge.

EXAMPLES.

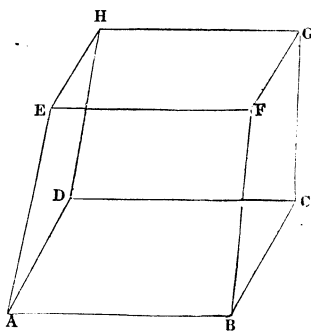
1. What is the solidity of a wedge, whose base is 10 inches long, 4 inches wide, whose height is 8 inches, and whose edge is 9 inches?
Ans. $154\frac{2}{3}$ cubic inches.

2. The length and breadth of the base of a wedge is 70 inches and 30 inches, the edge is 110 inches, and the altitude 42 inches. What is the solidity?
Ans. 30·3819 cubic feet.

PROBLEM VIII.

To find the volume of a rectangular prismoid.

Definition.—A rectangular prismoid takes its name from its semblance to a prism. It may also be said to resemble a frustum of a quadrangular pyramid. The upper and lower bases are rectangles, having their corresponding sides parallel, and the convex surface consists of four trapezoids. The altitude of the prismoid is the perpendicular distance between the bases.



Put $L = AB$, the length of lower base.

$B = AD$, the breadth of “ “

$l = EF$, the length of upper base.

$b = EH$, the breadth of “ “

h = the altitude of the prismoid.

If we pass a plane through EF and DC , the prismoid will be divided into two wedges, whose bases are the bases of the pris-

moid, and whose common altitude is the same as the altitude of the prismoid.

The solidity of these wedges will be (Problem VII.)

$$\frac{1}{6} (2L + l) Bh, \frac{1}{6} (2l + L) bh.$$

Adding, we have for solidity of prismoid,

$$\frac{1}{6} h (2BL + 2bl + Bl + bL). \quad . \quad . \quad . \quad (1.)$$

If we put $p = \frac{1}{2} (L + l)$; $q = \frac{1}{2} (B + b)$, we shall have

$$\text{solidity of prismoid} = \frac{1}{6} h (BL + bl + 4pq). \quad . \quad . \quad . \quad (2.)$$

In this last expression pq is the area of a parallel section equally distant from the two bases. Hence the following

RULE.

To the sum of the two bases add four times the area of a parallel section equally distant from the bases, and multiply the sum by $\frac{1}{6}$ of the height.

EXAMPLES.

1. How many cubic feet in a block of marble in the form of a rectangular prismoid, the lower base being 4 feet by 3, the upper base 3 feet by 2, and the height being 5 feet?

Ans. $44\frac{1}{6}$ cubic feet.

2. How many cubic feet in a stick 36 feet long, in the form of a rectangular prismoid, the one end being 20 inches by 16 inches, and the other end being 14 inches by 12 inches?

Ans. 60 cubic feet.

PROBLEM IX.

To find the inclination of two adjacent faces of a regular polyedron, and the radii of the inscribed and circumscribed spheres; also, the area of its surface, and its volume.

Let AB be the edge common to two adjacent faces, C and E the centres of those faces. Draw CO and EO perpendicular

to these faces, meeting at O; also draw CD, ED perpendicular to the common edge AB. The angle CDE will be the inclination of these faces.

Let I = inclination of the faces = angle CDE;

a = one of the edges = AB;

r = radius of the inscribed sphere
= OC = OE;

R = radius of the circumscribed sphere
= OA = OB;

m = number of faces forming a polyedral angle;

n = number of sides in each face;

f = whole number of faces in polyedron;

S = whole surface of polyedron;

V = the volume.

If, now, a sphere be described about O as a centre with a radius equal to unity, its intersection with the planes OAC, OAD, ODC will form a spherical triangle acd , right-angled at d . By *Napier's Rules*, this triangle gives

$$\cos. cd = \frac{\cos. cad}{\sin. acd}; \quad . \quad . \quad . \quad (1.)$$

$$\cos. ac = \cot. cad \cot. acd. \quad . \quad . \quad (2.)$$

Since cd measures the angle COD, and $COD = 90^\circ - \frac{1}{2}I$, we have $\cos. cd = \sin. \frac{1}{2}I$. We also have

$$\text{angle } cad = \frac{\pi}{m}; \quad \text{angle } acd = \frac{\pi}{n};$$

hence (1) becomes

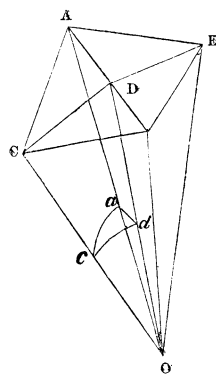
$$\sin. \frac{1}{2}I = \frac{\cos. \frac{\pi}{m}}{\sin. \frac{\pi}{n}}. \quad . \quad . \quad . \quad (3.)$$

The triangle ACD gives

$$CD = \frac{1}{2}a \cot. \frac{\pi}{n},$$

but triangle COD gives $r = CD \tan. \frac{1}{2}I$; hence we have

$$r = \frac{1}{2}a \tan. \frac{1}{2}I \cot. \frac{\pi}{n}. \quad . \quad . \quad . \quad (4.)$$



Triangle AOC gives $R \cos. AOC = R \cos. ac = r$; using the value of $\cos. ac$ given by (2), we find

$$R = r \tan. \frac{\pi}{m} \tan. \frac{\pi}{n} = \frac{1}{2} a \tan. \frac{1}{2} I \tan. \frac{\pi}{m}. \quad (5.)$$

The area of one of the faces is

$$n \times AD \times CD = n \times \frac{1}{2} a \times \frac{1}{2} a \cot. \frac{\pi}{n} = a^2 \times \frac{n}{4} \cot. \frac{\pi}{n};$$

consequently we have

$$S = a^2 \times \frac{nf}{4} \cot. \frac{\pi}{n}. \quad (6.)$$

Since the volume is equal to $S \times \frac{1}{3} r$, we find

$$V = a^3 \times \frac{nf}{24} \tan. \frac{1}{2} I \cot. \frac{\pi}{n}. \quad (7.)$$

These equations are general and apply to the five *Platonic* bodies. The values of m and n are as follows:

Tetraedron, $m = 3, n = 3$. Hexaedron, $m = 3, n = 4$.

Octaedron, $m = 4, n = 3$. Dodecaedron, $m = 3, n = 5$.

Icosaedron, $m = 5, n = 3$.

Using these values in (3), we find as follows:

$$\text{Tetraedron, } \sin. \frac{1}{2} I = \frac{\cos. 60^\circ}{\sin. 60^\circ} = \frac{1}{3} \sqrt{3}.$$

$$\text{Hexaedron, } \sin. \frac{1}{2} I = \frac{\cos. 60^\circ}{\sin. 45^\circ} = \frac{1}{2} \sqrt{2}.$$

$$\text{Octaedron, } \sin. \frac{1}{2} I = \frac{\cos. 45^\circ}{\sin. 60^\circ} = \frac{1}{3} \sqrt{6}.$$

$$\text{Dodecaedron, } \sin. \frac{1}{2} I = \frac{\cos. 60^\circ}{\sin. 36^\circ} = \frac{1}{10} \sqrt{50 + 10 \sqrt{5}}.$$

$$\text{Icosaedron, } \sin. \frac{1}{2} I = \frac{\cos. 36^\circ}{\sin. 60^\circ} = \frac{1}{10} (5 + \sqrt{5}).$$

Having thus found $\sin. \frac{1}{2} I$, we can find $\tan. \frac{1}{2} I$ by the formula

$$\tan. \frac{1}{2} I = \frac{\sin. \frac{1}{2} I}{\sqrt{1 - \sin.^2 \frac{1}{2} I}};$$

and then equations (4), (5), (6), and (7) will give, after considerable reduction, as follows:

$$\begin{aligned}
\text{Tetraedron, } & \begin{cases} r = a \times \frac{1}{\sqrt{2}} \sqrt{6}; & R = a \times \frac{1}{4} \sqrt{6}; \\ S = a^2 \times \sqrt{3}; & V = a^3 \times \frac{1}{\sqrt{2}} \sqrt{2}. \end{cases} \\
\text{Hexaedron, } & \begin{cases} r = a \times \frac{1}{2}; & R = a \times \frac{1}{2} \sqrt{2}; \\ S = a^2 \times 6; & V = a^3 \times 1. \end{cases} \\
\text{Octaedron, } & \begin{cases} r = a \times \frac{1}{6} \sqrt{6}; & R = a \times \frac{1}{2} \sqrt{2}; \\ S = a^2 \times 2 \sqrt{3}; & V = a^3 \times \frac{1}{3} \sqrt{2}. \end{cases} \\
\text{Dodecaedron, } & \begin{cases} r = a \times \frac{1}{\sqrt{6}} \sqrt{250 + 110\sqrt{5}}; & R = a \times \frac{1}{4} (\sqrt{15} + \sqrt{3}); \\ S = a^2 \times 3 \sqrt{25 + 10\sqrt{5}}; & V = a^3 \times \frac{1}{4} (15 + 7\sqrt{5}). \end{cases} \\
\text{Icosaedron, } & \begin{cases} r = a \times \frac{1}{\sqrt{2}} (3\sqrt{3} + \sqrt{15}); & R = a \times \frac{1}{4} \sqrt{10 + 2\sqrt{5}}; \\ S = a^2 \times 5 \sqrt{3}; & V = a^3 \times \frac{5}{\sqrt{2}} (3 + \sqrt{5}). \end{cases}
\end{aligned}$$

We give, in the next table, to seven decimal places, the values, and their logarithms, of the surfaces and volumes of the five polyedrons, having unity for the edges of each :

TABLE.

<i>Names.</i>	<i>No. of Faces.</i>	<i>Surfaces.</i>	<i>Volumes.</i>
Tetraedron . . .	4	1.7320508	0.1178513
		log. [0.2385607]	log. [1.0713344]
Hexaedron . . .	6	6.0000000	1.0000000
		log. [0.7781513]	log. [0.0000000]
Octaedron . . .	8	3.4641016	0.4714045
		log. [0.5395907]	log. [1.6733937]
Dodecaedron . . .	12	20.6457288	7.6631189
		log. [1.3148302]	log. [0.8844056]
Icosaedron . . .	20	8.6602540	2.1816950
		log. [0.9375307]	log. [0.3387940]

The numbers included within the brackets are the logarithms of those numbers immediately above them.

Since surfaces of similar solids are to each other as the squares of their like dimensions, we may find the surface of any regular polyedron by this

RULE.

Multiply the square of one of the edges by the tabular surface of the similar unit solid, as given in the foregoing table.

Or using logarithms.

To twice the logarithm of the edge add the tabular logarithm of the similar solid, as given in the table, and the sum will be the logarithm of the surface.

Again, since the volumes of similar solids are to each other as the cubes of their like dimensions, we may find the volume of any regular polyedron by this

RULE.

Multiply the cube of one of the edges by the tabular volume of the similar unit solid, as given in the table.

Or using logarithms.

To three times the logarithm of the edge add the tabular logarithm of the similar solid, as given in the table, and the sum will be the logarithm of the volume.

EXAMPLES.

1. What is the surface and the volume of a regular icosaedron, each edge of which is 10 inches?

$$\text{Ans. } \left\{ \begin{array}{l} \text{Surface} = 866.025 \text{ sq. inches.} \\ \text{Volume} = 2181.695 \text{ cubic inches.} \end{array} \right.$$

2. What is the surface and the volume of a regular dodecaedron, each edge of which is 3 feet?

$$\text{Ans. } \left\{ \begin{array}{l} \text{Surface} = 185.811 \text{ sq. feet.} \\ \text{Volume} = 206.902 \text{ cubic feet.} \end{array} \right.$$

3. What is the surface and the volume of a regular tetraedron, each edge being 4 inches?

$$\text{Ans. } \left\{ \begin{array}{l} \text{Surface} = 27.712 \text{ sq. inches.} \\ \text{Volume} = 7.542 \text{ cubic inches.} \end{array} \right.$$

PROBLEM X.

To find the surface and volume of any parallelopipedon.

Let DE be a parallelopipedon, having the edges

$$DB = a,$$

$$DC = b,$$

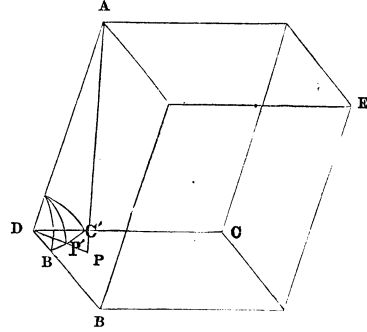
$$DA = c,$$

and their inclinations

$$BDC = \alpha,$$

$$CDA = \beta,$$

$$ADB = \gamma, \text{ given.}$$



The area of any face, as the face BC, $= ab \sin. \alpha$; hence, for the whole surface, we have

$$S = (ab \sin. \alpha + bc \sin. \beta + ca \sin. \gamma). \quad (1.)$$

To find the volume, draw AP perpendicular to BC, and pass a plane through AD and AP, intersecting BC in DP. Conceive the surface of a sphere whose radius is unity to be described about D as a centre, and the arcs formed by its intersection with the planes passing through the centre D will be as follows:

$$B'C' = \alpha, \quad C'A' = \beta, \quad A'B' = \gamma.$$

The arc A'P' measures the angle ADP.

The altitude $AP = AD \sin. ADP = c \sin. A'P'$; this, multiplied by the base $BC = ab \sin. \alpha$, gives, for the volume,

$$V = abc \sin. \alpha \sin. A'P'. \quad (2.)$$

The right spherical triangle A'B'P' gives

$$\sin. A'P' = \sin. \gamma \sin. A'B'P';$$

hence (2) becomes

$$V = abc \sin. \alpha \sin. \gamma \sin. A'B'P'. \quad (3.)$$

In the spherical triangle A'B'C' we have, by formula (C), § 57, if $s = \frac{1}{2}(\alpha + \beta + \gamma)$,

$$\sin. A'B'P' = \frac{2}{\sin. \alpha \sin. \gamma} \sqrt{\sin. s \sin. (s - \alpha) \sin. (s - \beta) \sin. (s - \gamma)};$$

hence (3) becomes

$$V = 2abc \sqrt{\sin. s \sin. (s - \alpha) \sin. (s - \beta) \sin. (s - \gamma)}. \quad (4.)$$

Since the volume of the tetraedron formed by passing a plane through the three points ABC is one sixth of the parallelepipedon, it follows, that if a, b, c denote the three edges of a polyedral angle of a tetraedron, and α, β, γ are their inclinations, the volume of the tetraedron will be

$$V = \frac{1}{6} abc \sqrt{\sin. s \sin. (s - \alpha) \sin. (s - \beta) \sin. (s - \gamma)}. \quad (5.)$$

PROBLEM XI.

To find the surface of a sphere.

RULE.

Multiply the square of the diameter by $\pi = 3.1415926$. [See Geom., B. VII., T. XII., S.]

EXAMPLES.

1. Required the number of square inches in the surface of a ball 6 inches in diameter. *Ans.* 113.0976 sq. inches.
2. How many square miles on the surface of the earth, it being 7912 miles in diameter. *Ans.* 196662896.

PROBLEM XII.

To find the volume of a sphere.

RULE.

Multiply the cube of the diameter by $\frac{1}{6} \pi = 0.5235988$. [See Geom., B. VII., T. XII., S.]

EXAMPLES.

1. What is the volume of a sphere 6 inches in diameter? *Ans.* 113.0976 cubic inches.
2. How many cubic miles in the earth, its diameter being 7912 miles? *Ans.* 259332805350 cubic miles.

PROBLEM XIII.

To find the area of a portion of the surface of a sphere included between three arcs of great circles. That is, to find the area of a spherical triangle.

RULE.

From the sum of the three angles of the triangle subtract 180° , divide the remainder by 90° and multiply the quotient by one-eighth of the entire surface of the sphere of which the triangle forms a part. [Geom. B. VIII., T. XIX., S.]

EXAMPLES.

1. How many square inches in a spherical triangle, described on a sphere 6 inches in diameter, and whose angles are 90° , 100° , 110° ? *Ans.* 47.124 sq. inches.

2. What part of the entire surface of a sphere is the spherical triangle, each of whose angles is 120° ? *Ans.* $\frac{1}{4}$.

§ 88. The following multipliers with their logarithms will be very convenient for reference.

TABLE OF MULTIPLIERS.

	<i>Log. of multipliers.</i>
1. Radius of a circle $\times 6.28318531 =$ Circumference.....	.07981799
2. Square of the radius of a circle $\times 3.14159265 =$ Area.....	.04971499
3. Diameter of a circle $\times 3.14159265 =$ Circumference.....	.04971499
4. Square of the diameter of a circle $\times 0.78539816 =$ Area.....	.18950899
5. Circumference of a circle $\times 0.15915494 =$ Radius.....	.12018201
6. Circumference of a circle $\times 0.31830989 =$ Diameter.....	.15028501
7. Square root of area of a circle $\times 0.56418958 =$ Radius.....	.17514251
8. Square root of area of a circle $\times 1.12837917 =$ Diameter.....	.00524551
9. Radius of circle $\times 1.73205081 =$ Side of inscribed equilateral triangle.....	.02385606
10. Side of inscribed equilateral triangle $\times 0.57735027 =$ Radius of circle.....	.17614394
11. Radius of a circle $\times 1.41421356 =$ Side of inscribed square.....	.01505150
12. Side of inscribed square $\times 0.70710678 =$ Radius.....	.18494850
13. Square of radius of a sphere $\times 12.56637061 =$ Surface.....	.10992099
14. Cube of radius of a sphere $\times 4.18879020 =$ Volume.....	.06220886
15. Square of diameter of a sphere $\times 3.14159265 =$ Surface.....	.04971499
16. Cube of diameter of a sphere $\times 0.52359878 =$ Volume.....	.17189986
17. Square of circumference of a sphere $\times 0.31830989 =$ Surface.....	.15028501
18. Cube of circumference of a sphere $\times 0.01688686 =$ Volume.....	.22275490
19. Square root of surface of a sphere $\times 0.28209479 =$ Radius.....	.14503951
20. Square root of surface of a sphere $\times 0.56418958 =$ Diameter.....	.17514251
21. Square root of surface of a sphere $\times 1.77245385 =$ Circumference.....	.02485748
22. Cube root of volume of a sphere $\times 0.62035049 =$ Radius.....	.17926371
23. Cube root of volume of a sphere $\times 1.24070098 =$ Diameter.....	.00936671
24. Cube root of volume of a sphere $\times 3.89777707 =$ Circumference.....	.05908170
25. Radius of a sphere $\times 1.15470054 =$ Side of inscribed cube.....	.00624694
26. Side of inscribed cube $\times 0.86602540 =$ Radius.....	.19375306

TABLE I.,

OF

LOGARITHMS OF NUMBERS

FROM

1 TO 10000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0.000000	26	1.414973	51	1.707570	76	1.880814
2	0.301030	27	1.431364	52	1.716003	77	1.886491
3	0.477121	28	1.447158	53	1.724276	78	1.892095
4	0.602060	29	1.462398	54	1.732394	79	1.897627
5	0.698970	30	1.477121	55	1.740363	80	1.903090
6	0.778151	31	1.491362	56	1.748188	81	1.908485
7	0.845098	32	1.505130	57	1.755875	82	1.913814
8	0.903090	33	1.518514	58	1.763428	83	1.919078
9	0.954243	34	1.531479	59	1.770852	84	1.924279
10	1.000000	35	1.544068	60	1.778151	85	1.929419
11	1.041393	36	1.556303	61	1.785330	86	1.934498
12	1.079181	37	1.568202	62	1.792392	87	1.939510
13	1.113943	38	1.579784	63	1.799341	88	1.944483
14	1.146128	39	1.591065	64	1.806180	89	1.949390
15	1.176091	40	1.602060	65	1.812913	90	1.954243
16	1.204120	41	1.612784	66	1.819544	91	1.959041
17	1.230449	42	1.623240	67	1.826075	92	1.963788
18	1.255273	43	1.633468	68	1.832509	93	1.968483
19	1.278754	44	1.643453	69	1.838849	94	1.973128
20	1.301030	45	1.653213	70	1.845098	95	1.977724
21	1.322219	46	1.662758	71	1.851258	96	1.982271
22	1.342423	47	1.672098	72	1.857333	97	1.986772
23	1.361728	48	1.681241	73	1.863323	98	1.991226
24	1.380211	49	1.690196	74	1.869232	99	1.995635
25	1.397940	50	1.698970	75	1.875061	100	2.000000

N. B. In the following table, in the last nine columns of each page, where the first or leading figures change from 9's to 0's, the character * is introduced instead of the 0's, to catch the eye, and to indicate that from thence the annexed first two figures of the Logarithm in the second column stand in the next lower line, directly under the *asterisk*.

N.	0	1	2	3	4	5	6	7	8	9	D.
100	00 0000	0434	0868	1301	1734	2166	2598	3029	3461	3891	432
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
102	* 8600	9026	9451	9876	*300	0724	1147	1570	1993	2415	424
103	01 2837	3259	3680	4100	4521	4940	5360	5779	6197	6616	419
104	* 7033	7451	7868	8284	8700	9116	9532	9947	*361	0775	416
105	02 1189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
107	* 9384	9789	*195	0600	1004	1408	1812	2216	2619	3021	404
108	03 3424	3826	4227	4628	5029	5430	5830	6230	6629	7028	400
109	* 7426	7825	8223	8620	9017	9414	9811	*207	0602	0998	396
110	04 1393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	389
112	* 9218	9606	9993	*380	0766	1153	1538	1924	2309	2694	386
113	05 3078	3463	3846	4230	4613	4996	5378	5760	6142	6524	382
114	* 6905	7286	7666	8046	8426	8805	9185	9563	9942	*320	379
115	06 0698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	372
117	* 8186	8557	8928	9298	9668	*038	0407	0776	1145	1514	369
118	07 1882	2250	2617	2985	3352	3718	4085	4451	4816	5182	366
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	* 9181	9543	9904	*266	0626	0987	1347	1707	2067	2426	360
121	08 2785	3144	3503	3861	4219	4576	4934	5291	5647	6004	357
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
123	* 9905	*258	0611	0963	1315	1667	2018	2370	2721	3071	351
124	09 3422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	* 6910	7257	7604	7951	8298	8644	8990	9335	9681	*026	346
126	10 0371	0715	1059	1403	1747	2091	2434	2777	3119	3462	343
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	340
128	* 7210	7549	7888	8227	8565	8903	9241	9579	9916	*253	338
129	11 0590	0926	1263	1599	1934	2270	2605	2940	3275	3609	335
130	3043	4277	4611	4944	5278	5611	5943	6276	6608	6940	333
131	* 7271	7603	7934	8265	8595	8926	9256	9586	9915	*245	330
132	12 0574	0903	1231	1560	1888	2216	2544	2871	3198	3525	328
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
134	* 7105	7429	7753	8076	8399	8722	9045	9368	9690	*012	323
135	13 0334	0655	0977	1298	1619	1939	2260	2580	2900	3219	321
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	315
138	* 9879	*194	0508	0822	1136	1450	1763	2076	2389	2702	314
139	14 3015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	6128	6438	6748	7058	7367	7676	7985	8294	8603	8911	309
141	* 9219	9527	9835	*142	0449	0756	1063	1370	1676	1982	307
142	15 2288	2594	2900	3205	3510	3815	4120	4424	4728	5032	305
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
144	* 8362	8664	8965	9266	9567	9868	*168	0469	0769	1068	301
145	16 1368	1667	1967	2266	2564	2863	3161	3460	3758	4055	299
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
148	17 0262	0555	0848	1141	1434	1726	2019	2311	2603	2895	293
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
150	6091	6381	6670	6959	7248	7536	7825	8113	8401	8689	289
151	* 8977	9264	9552	9839	*126	0413	0699	0985	1272	1558	287
152	18 1844	2129	2415	2700	2985	3270	3555	3839	4123	4407	285
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
154	* 7521	7803	8084	8366	8647	8928	9209	9490	9771	*051	281
155	19 0332	0612	0892	1171	1451	1730	2010	2289	2567	2846	279
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
157	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
158	* 8657	8932	9206	9481	9755	*029	0303	0577	0850	1124	274
159	20 1397	1670	1943	2216	2488	2761	3033	3305	3577	3848	272
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TABLE I.

LOGARITHMS OF NUMBERS.

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161	6826	7099	7365	7634	7904	8173	8441	8710	8979	9247	269
162	* 9515	9783	* 051	0319	0586	0853	1121	1388	1654	1921	267
163	21 2188	2454	2720	2986	3252	3518	3783	4049	4314	4579	266
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
166	22 0108	0370	0631	0892	1153	1414	1675	1936	2196	2456	261
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
169	* 7887	8144	8400	8657	8913	9170	9426	9682	9938	* 193	256
170	23 0449	0704	0960	1215	1470	1724	1979	2234	2488	2742	254
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
173	* 8046	8297	8548	8799	9049	9299	9550	9800	* 050	0300	250
174	24 0549	0799	1048	1297	1546	1795	2044	2293	2541	2790	249
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
177	* 7973	8219	8464	8709	8954	9198	9443	9687	9932	* 176	245
178	25 0430	0664	0908	1151	1395	1638	1881	2125	2368	2610	243
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	5273	5514	5755	5996	6237	6477	6718	6958	7198	7439	241
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
182	26 0071	0310	0548	0787	1025	1263	1501	1739	1976	2214	238
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
186	* 0513	0746	0980	* 213	0446	0679	0912	1144	1377	1609	233
187	27 1842	2074	2306	2538	2770	3001	3233	3464	3696	3927	232
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
190	* 8754	8982	9211	9439	9667	9895	* 123	0351	0578	0806	228
191	28 1033	1261	1488	1715	1942	2169	2396	2622	2849	3075	227
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
195	29 0035	0257	0480	0702	0925	1147	1369	1591	1813	2034	222
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
199	* 8853	9071	9289	9507	9725	9943	* 161	0378	0595	0813	218
200	30 1030	1247	1464	1681	1898	2114	2331	2547	2764	2980	217
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
204	* 9630	9843	* 056	0268	0481	0693	0906	1118	1330	1542	212
205	31 1754	1966	2177	2389	2600	2812	3023	3234	3445	3656	211
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
209	32 0146	0354	0562	0769	0977	1184	1391	1598	1805	2012	207
210	2219	2426	2633	2839	3046	3252	3458	3665	3871	4077	206
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
213	* 8380	8583	8787	8991	9194	9398	9601	9805	* 008	0211	203
214	33 0414	0617	0819	1022	1225	1427	1630	1832	2034	2236	202
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
218	* 8456	8656	8855	9054	9253	9451	9650	9849	* 047	0246	199
219	34 0444	0642	0841	1039	1237	1435	1632	1830	2028	2225	198
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220	34 2423	2620	2817	3014	3212	3409	3606	3802	3999	4196	197
221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
222	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
223	* 8305	8500	8694	8889	9083	9278	9472	9666	9860	+034	194
224	35 0248	0442	0636	0829	1023	1216	1410	1603	1796	1989	193
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	193
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
229	* 9835	+025	0215	0404	0593	0783	0972	1161	1350	1539	189
230	36 1728	1917	2105	2294	2482	2671	2859	3048	3236	3424	188
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
234	* 9216	9401	9587	9772	9958	+143	0328	0513	0698	0883	185
235	37 1068	1253	1437	1622	1806	1991	2175	2360	2544	2728	184
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
239	* 8398	8580	8761	8943	9124	9306	9487	9668	9849	+030	181
240	38 0211	0392	0573	0754	0934	1115	1296	1476	1656	1837	181
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
244	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989	178
245	* 9166	9343	9520	9698	9875	+051	0228	0405	0582	0759	177
246	39 0935	1112	1288	1464	1641	1817	1993	2169	2345	2521	176
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
248	4432	4607	4782	4957	5132	5306	5481	5656	5830	6005	175
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
250	7940	8114	8287	8461	8634	8808	8981	9154	9328	9501	173
251	* 9674	9847	+020	0192	0365	0538	0711	0883	1056	1228	173
252	40 1401	1573	1745	1917	2089	2261	2433	2605	2777	2949	172
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	170
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
257	* 9933	+102	0271	0440	0609	0777	0946	1114	1283	1451	169
258	41 1620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
260	4973	5140	5307	5474	5641	5808	5974	6141	6308	6474	167
261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
262	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	165
263	* 9956	+121	0286	0451	0616	0781	0945	1110	1275	1439	165
264	42 1604	1768	1933	2097	2261	2426	2590	2754	2918	3082	164
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	164
266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
268	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	162
269	* 9752	9914	+075	0236	0398	0559	0720	0881	1042	1203	161
270	43 1364	1525	1685	1846	2007	2167	2328	2488	2649	2809	161
271	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	160
272	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
273	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592	159
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	158
275	* 9333	9491	9648	9806	9964	+122	0279	0437	0594	0752	158
276	44 0909	1066	1224	1381	1538	1695	1852	2009	2166	2323	157
277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	157
278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
279	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	155
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TABLE I.

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280	44 7158	7313	7468	7623	7778	7933	8088	8242	8397	8552	155
281	* 8706	8861	9015	9170	9324	9478	9633	9787	9941	•095	154
282	45 0249	0403	0557	0711	0865	1018	1172	1326	1479	1633	154
283	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	153
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
288	* 9392	9543	9694	9845	9995	+146	0296	0447	0597	0748	151
289	46 0898	1048	1198	1348	1499	1649	1799	1948	2098	2248	150
290	2398	2548	2697	2847	2997	3146	3296	3445	3594	3744	150
291	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	149
293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
294	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	148
295	* 9822	9969	+116	0263	0410	0557	0704	0851	0998	1145	147
296	47 1292	1438	1585	1732	1878	2025	2171	2318	2464	2610	146
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	146
299	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145
300	7121	7266	7411	7555	7700	7844	7989	8133	8278	8422	145
301	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863	144
302	48 0007	0151	0294	0438	0582	0725	0869	1012	1156	1299	144
303	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
304	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157	143
305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
306	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	142
307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
308	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818	141
309	* 9958	+099	0239	0380	0520	0661	0801	0941	1081	1222	140
310	49 1362	1502	1642	1782	1922	2062	2201	2341	2481	2621	140
311	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015	139
312	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406	139
313	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	139
314	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	138
315	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550	138
316	* 9687	9824	9962	+099	0236	0374	0511	0648	0785	0922	137
317	50 1059	1196	1333	1470	1607	1744	1880	2017	2154	2291	137
318	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655	136
319	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014	136
320	5150	5286	5421	5557	5693	5828	5964	6099	6234	6370	136
321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721	135
322	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068	135
323	* 9203	9337	9471	9606	9740	9874	+009	0143	0277	0411	134
324	51 0345	0679	0813	0947	1081	1215	1349	1482	1616	1750	134
325	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084	133
326	3218	3351	3484	3617	3750	3883	4016	4149	4282	4414	133
327	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	133
328	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064	132
329	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382	132
330	8514	8646	8777	8909	9040	9171	9303	9434	9566	9697	131
331	* 9828	9959	+090	0221	0353	0484	0615	0745	0876	1007	131
332	52 1138	1269	1400	1530	1661	1792	1922	2053	2183	2314	131
333	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616	130
334	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915	130
335	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210	129
336	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501	129
337	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788	129
338	* 8917	9045	9174	9302	9430	9559	9687	9815	9943	+072	128
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341	2754	2882	3009	3136	3264	3391	3518	3645	3772	3899	127
342	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167	127
343	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432	126
344	6558	6685	6811	6937	7063	7189	7315	7441	7567	7693	126
345	7819	7945	8071	8197	8322	8448	8574	8699	8825	8951	126
346	* 9076	9202	9327	9452	9578	9703	9829	9954	• 079	0204	125
347	54 0329	0455	0580	0705	0830	0955	1080	1205	1330	1454	125
348	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701	125
349	2825	2950	3074	3199	3323	3447	3571	3696	3820	3944	124
350	4068	4192	4316	4440	4564	4688	4812	4936	5060	5183	124
351	5307	5431	5555	5678	5802	5925	6049	6172	6296	6419	124
352	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652	123
353	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881	123
354	* 9003	9126	9249	9371	9494	9616	9739	9861	9984	• 106	123
355	55 0228	0351	0473	0595	0717	0840	0962	1084	1206	1328	122
356	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547	122
357	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762	121
358	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973	121
359	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182	121
360	6303	6423	6544	6664	6785	6905	7026	7146	7267	7387	120
361	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589	120
362	8709	8829	8948	9068	9188	9308	9428	9548	9667	9787	120
363	* 9907	• 026	0146	0265	0385	0504	0624	0743	0863	0982	119
364	56 1101	1221	1340	1459	1578	1698	1817	1936	2055	2174	119
365	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362	119
366	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	119
367	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
368	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909	118
369	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	118
370	8202	8319	8436	8554	8671	8788	8905	9023	9140	9257	117
371	* 9374	9491	9608	9725	9842	9959	• 076	0193	0309	0426	117
372	57 0543	0660	0776	0893	1010	1126	1243	1359	1476	1592	117
373	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
374	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915	116
375	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072	116
376	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226	115
377	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377	115
378	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525	115
379	8639	8754	8868	8983	9097	9212	9326	9441	9555	9669	114
380	* 9784	9898	• 012	0126	0241	0355	0469	0583	0697	0811	114
381	58 0925	1039	1153	1267	1381	1495	1608	1722	1836	1950	114
382	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085	114
383	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218	113
384	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348	113
385	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475	113
386	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599	112
387	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720	112
388	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838	112
389	* 9950	• 001	0173	0284	0396	0507	0619	0730	0842	0953	112
390	59 1065	1176	1287	1399	1510	1621	1732	1843	1955	2066	111
391	2177	2288	2399	2510	2621	2732	2843	2954	3064	3175	111
392	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282	111
393	4393	4503	4614	4724	4834	4945	5055	5165	5275	5386	110
394	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487	110
395	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586	110
396	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681	110
397	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774	109
398	* 9883	9992	• 101	0210	0319	0428	0537	0646	0755	0864	109
399	60 0973	1082	1191	1299	1408	1517	1625	1734	1843	1951	109
N.	0	1	2	3	4	5	6	7	8	9	D.

TABLE I.

LOGARITHMS OF NUMBERS.

7

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400	60 2060	2169	2277	2386	2494	2603	2711	2819	2928	3036	108
401	3144	3253	3361	3469	3577	3686	3794	3902	4010	4118	108
402	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197	108
403	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274	108
404	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348	107
405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419	107
406	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488	107
407	* 9594	9701	9808	9914	•021	0128	0234	0341	0447	0554	107
408	61 0600	0767	0873	0979	1086	1192	1298	1405	1511	1617	106
409	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678	106
410	2784	2890	2996	3102	3207	3313	3419	3525	3630	3736	106
411	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792	106
412	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845	105
413	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895	105
414	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943	105
415	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989	105
416	* 9093	9198	9302	9406	9511	9615	9719	9824	9928	•032	104
417	62 0136	0240	0344	0448	0552	0656	0760	0864	0968	1072	104
418	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	104
419	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146	104
420	3249	3353	3456	3559	3663	3766	3869	3973	4076	4179	103
421	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210	103
422	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238	103
423	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	103
424	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287	102
425	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308	102
426	* 9410	9512	9613	9715	9817	9919	•021	0123	0224	0326	102
427	63 0428	0530	0631	0733	0835	0936	1038	1139	1241	1342	102
428	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356	101
429	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	101
430	3468	3569	3670	3771	3872	3973	4074	4175	4276	4376	100
431	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383	100
432	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388	100
433	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390	100
434	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389	99
435	8489	8589	8689	8789	8888	8988	9088	9188	9287	9387	99
436	* 9486	9586	9686	9785	9885	9984	•084	0183	0283	0382	99
437	64 0481	0581	0680	0779	0879	0978	1077	1177	1276	1375	99
438	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366	99
439	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354	99
440	3453	3551	3650	3749	3847	3946	4044	4143	4242	4340	98
441	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324	98
442	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306	98
443	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	98
444	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	98
445	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	97
446	* 9335	9432	9530	9627	9724	9821	9919	•016	0113	0210	97
447	65 0308	0405	0502	0599	0696	0793	0890	0987	1084	1181	97
448	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150	97
449	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	97
450	3213	3309	3405	3502	3598	3695	3791	3888	3984	4080	96
451	4177	4273	4369	4465	4562	4658	4754	4850	4946	5042	96
452	5138	5235	5331	5427	5523	5619	5715	5810	5906	6002	96
453	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960	96
454	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	96
455	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870	95
456	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821	95
457	* 9916	•011	0106	0201	0296	0391	0486	0581	0676	0771	95
458	66 0865	0960	1055	1150	1245	1339	1434	1529	1623	1718	95
459	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663	95
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460	66 2758	2852	2947	3041	3135	3230	3324	3418	3512	3607	94
461	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548	94
462	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487	94
463	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424	94
464	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360	94
465	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293	93
466	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224	93
467	* 6317	9410	9503	9596	9689	9782	9875	9967	+060	0153	93
468	67 0246	0339	0431	0524	0617	0710	0802	0895	0988	1080	93
469	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005	93
470	2098	2190	2283	2375	2467	2560	2652	2744	2836	2929	92
471	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850	92
472	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769	92
473	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687	92
474	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	92
475	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	91
476	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427	91
477	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337	91
478	* 9428	9519	9610	9700	9791	9882	9973	+063	0154	0245	91
479	68 0336	0426	0517	0607	0698	0789	0879	0970	1060	1151	91
480	1241	1332	1422	1513	1603	1693	1784	1874	1964	2055	90
481	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957	90
482	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857	90
483	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756	90
484	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	90
485	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547	89
486	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	89
487	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	89
488	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	89
489	* 9309	9398	9486	9575	9664	9753	9841	9930	+019	0107	89
490	69 0196	0285	0373	0462	0550	0639	0728	0816	0905	0993	89
491	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	88
492	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	88
493	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639	88
494	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	88
495	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394	88
496	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	87
497	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	87
498	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	87
499	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883	87
500	8970	9057	9144	9231	9317	9404	9491	9578	9664	9751	87
501	* 9838	9924	+011	0098	0184	0271	0358	0444	0531	0617	87
502	70 0704	0790	0877	0963	1050	1136	1222	1309	1395	1482	86
503	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	86
504	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205	86
505	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065	86
506	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	86
507	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	86
508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	85
509	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	85
510	7570	7655	7740	7826	7911	7996	8081	8166	8251	8336	85
511	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185	85
512	* 9270	9355	9440	9524	9609	9694	9779	9863	9948	+033	85
513	71 0117	0202	0287	0371	0456	0540	0625	0710	0794	0879	85
514	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723	84
515	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566	84
516	2650	2734	2818	2902	2986	3070	3154	3238	3322	3407	84
517	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	84
518	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	84
519	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920	84
N.	0	1	2	3	4	5	6	7	8	9	D.

TABLE I.

LOGARITHMS OF NUMBERS.

9

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520	71 6003	6087	6170	6254	6337	6421	6504	6588	6671	6754	83
521	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587	83
522	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	83
523	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	83
524	* 9331	9414	9497	9580	9663	9745	9828	9911	9994	* 077	83
525	72 0159	0242	0325	0407	0490	0573	0655	0738	0821	0903	83
526	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	82
527	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	82
528	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	82
529	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	82
530	4276	4358	4440	4522	4604	4685	4767	4849	4931	5013	82
531	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	82
532	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	82
533	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	81
534	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	81
535	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084	81
536	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	81
537	* 9974	* 055	0136	0217	0298	0378	0459	0540	0621	0702	81
538	73 0782	0863	0944	1024	1105	1186	1266	1347	1428	1508	81
539	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	81
540	2394	2474	2555	2635	2715	2796	2876	2956	3037	3117	80
541	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919	80
542	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720	80
543	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519	80
544	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317	80
545	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113	80
546	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
547	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
548	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
549	* 9572	9651	9731	9810	9889	9968	* 047	0126	0205	0284	79
550	74 0363	0442	0521	0600	0678	0757	0836	0915	0994	1073	79
551	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	79
552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2646	79
553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
554	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	78
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
558	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	78
559	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	78
560	8188	8266	8343	8421	8498	8576	8653	8731	8808	8885	77
561	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	77
562	* 9736	9814	9891	9968	* 045	0123	0200	0277	0354	0431	77
563	75 0508	0586	0663	0740	0817	0894	0971	1048	1125	1202	77
564	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	77
565	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
566	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	77
567	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	77
568	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
569	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	76
570	5875	5951	6027	6103	6180	6256	6332	6408	6484	6560	76
571	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	76
572	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
573	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	76
574	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	76
575	* 9668	9743	9819	9894	9970	* 045	0121	0196	0272	0347	75
576	76 0422	0498	0573	0649	0724	0799	0875	0950	1025	1101	75
577	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	75
578	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	75
579	2679	2754	2829	2904	2978	3053	3128	3203	3278	3353	75
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580	76 3428	3503	3578	3653	3727	3802	3877	3952	4027	4101	75
581	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848	75
582	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	75
583	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	74
584	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	74
585	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	74
586	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
587	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	74
588	* 9377	9451	9525	9599	9673	9746	9820	9894	9968	* 042	74
589	77 0115	0189	0263	0336	0410	0484	0557	0631	0705	0778	74
590	0852	0926	0999	1073	1146	1220	1293	1367	1440	1514	74
591	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
592	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
593	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
594	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
595	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	73
596	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
597	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	73
598	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
599	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72
600	8151	8224	8296	8368	8441	8513	8585	8658	8730	8802	72
601	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524	72
602	* 9596	9669	9741	9813	9885	9957	* 0029	0101	0173	0245	72
603	78 0317	0389	0461	0533	0605	0677	0749	0821	0893	0965	72
604	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	72
605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	72
606	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	72
607	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	71
608	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	71
609	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	71
610	5330	5401	5472	5543	5615	5686	5757	5828	5899	5970	71
611	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
612	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	71
613	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	71
614	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	71
615	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	71
616	* 9581	9651	9722	9792	9863	9933	* 0004	0074	0144	0215	70
617	79 0285	0356	0426	0496	0567	0637	0707	0778	0848	0918	70
618	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	70
619	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	70
620	2392	2462	2532	2602	2672	2742	2812	2882	2952	3022	70
621	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721	70
622	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	70
623	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	70
624	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	70
625	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	69
626	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
627	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
628	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	69
629	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69
630	9341	9409	9478	9547	9616	9685	9754	9823	9892	9961	69
631	80 0029	0098	0167	0236	0305	0373	0442	0511	0580	0648	69
632	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	69
633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
634	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	69
635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
636	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
637	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
639	5501	5569	5637	5705	5773	5841	5909	5977	6044	6112	68
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TABLE I.

LOGARITHMS OF NUMBERS.

11

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640	80 6180	6248	6316	6384	6451	6519	6587	6655	6723	6790	68
641	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	68
642	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	68
643	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	67
644	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	67
645	* 9560	9627	9694	9762	9829	9896	9964	* 031	0098	0165	67
646	81 0233	0300	0367	0434	0501	0569	0636	0703	0770	0837	67
647	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
648	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
649	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	67
650	2913	2980	3047	3114	3181	3247	3314	3381	3448	3514	67
651	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	67
652	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	67
653	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
654	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
655	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	66
656	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	66
657	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	66
658	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	66
659	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	66
660	* 9544	9610	9676	9741	9807	9873	9939	* 004	0070	0136	66
661	82 0201	0267	0333	0399	0464	0530	0595	0661	0727	0792	66
662	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448	66
663	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	65
664	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	65
665	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	65
666	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	65
667	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	65
668	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	65
669	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	65
670	6075	6140	6204	6269	6334	6399	6464	6528	6593	6658	65
671	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	65
672	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	65
673	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	64
674	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	64
675	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	64
676	* 9947	* 011	0075	0139	0204	0268	0332	0396	0460	0525	64
677	83 0589	0653	0717	0781	0845	0909	0973	1037	1102	1166	64
678	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	64
679	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	64
680	2509	2573	2637	2700	2764	2828	2892	2956	3020	3083	64
681	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	64
682	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	64
683	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	64
684	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	63
685	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	63
686	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	63
687	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	63
688	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	63
689	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
690	8849	8912	8975	9038	9101	9164	9227	9290	9352	9415	63
691	* 9478	9541	9604	9667	9729	9792	9855	9918	9981	* 043	63
692	84 0106	0169	0232	0294	0357	0420	0482	0545	0608	0671	63
693	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297	63
694	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	63
695	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	62
696	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	62
697	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	62
698	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	62
699	4477	4539	4601	4664	4726	4788	4850	4912	4974	5036	62
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700	84 5098	5160	5222	5284	5346	5408	5470	5532	5594	5656	62
701	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	62
702	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	62
703	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	62
704	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	62
705	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	62
706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
707	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	61
708	85 0033	0095	0156	0217	0279	0340	0401	0462	0524	0585	61
709	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	61
710	1258	1320	1381	1442	1503	1564	1625	1686	1747	1809	61
711	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419	61
712	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029	61
713	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	61
714	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	61
715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	61
716	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	61
717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	61
718	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	60
719	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	60
720	7332	7393	7453	7513	7574	7634	7694	7755	7815	7875	60
721	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	60
722	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	60
723	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679	60
724	* 9739	9799	9859	9918	9978	+038	0098	0158	0218	0278	60
725	86 0338	0398	0458	0518	0578	0637	0697	0757	0817	0877	60
726	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	60
727	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	60
728	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	60
729	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	60
730	3323	3382	3442	3501	3561	3620	3680	3739	3799	3858	59
731	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	59
732	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	59
733	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	59
734	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	59
735	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	59
736	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	59
737	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	59
738	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	59
739	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	59
740	9232	9290	9349	9408	9466	9525	9584	9642	9701	9760	59
741	* 9818	9877	9935	9994	+053	0111	0170	0228	0287	0345	59
742	87 0404	0462	0521	0579	0638	0696	0755	0813	0872	0930	58
743	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	58
744	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	58
745	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	58
746	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	58
747	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	58
748	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	58
749	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	58
750	5061	5119	5177	5235	5293	5351	5409	5466	5524	5582	58
751	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	58
752	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	58
753	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	58
754	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	58
755	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464	57
756	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	57
757	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	57
758	* 9669	9726	9784	9841	9898	9956	+013	0070	0127	0185	57
759	88 0242	0299	0356	0413	0471	0528	0585	0642	0699	0756	57
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TABLE I.

LOGARITHMS OF NUMBERS.

13

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760	88 0814	0871	0928	0985	1042	1099	1156	1213	1271	1328	57
761	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898	57
762	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	57
763	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	57
764	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	57
765	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172	57
766	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	57
767	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	57
768	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	57
769	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	56
770	6491	6547	6604	6660	6716	6773	6829	6885	6942	6998	56
771	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	56
772	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	56
773	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	56
774	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	56
775	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	56
776	* 9862	9918	9974	+030	0086	0141	0197	0253	0309	0365	56
777	89 0421	0477	0533	0589	0645	0700	0756	0812	0868	0924	56
778	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	56
779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	56
780	2095	2150	2206	2262	2317	2373	2429	2484	2540	2595	56
781	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	56
782	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	56
783	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	55
784	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	55
785	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	55
786	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
787	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	55
788	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	55
789	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
790	7627	7682	7737	7792	7847	7902	7957	8012	8067	8122	55
791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	55
792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	55
793	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	55
794	* 9821	9875	9930	9985	+039	0094	0149	0203	0258	0312	55
795	90 0367	0422	0476	0531	0586	0640	0695	0749	0804	0859	55
796	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	55
797	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	54
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
799	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	54
800	3090	3144	3199	3253	3307	3361	3416	3470	3524	3578	54
801	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	54
802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	54
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	54
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	54
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	54
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
810	8485	8539	8592	8646	8699	8753	8807	8860	8914	8967	54
811	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	54
812	* 9556	9610	9663	9716	9770	9823	9877	9930	9984	+037	53
813	91 0091	0144	0197	0251	0304	0358	0411	0464	0518	0571	53
814	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	53
815	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	53
816	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
817	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
819	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761	53
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
820	91 3814	3867	3920	3973	4026	4079	4132	4184	4237	4290	53
821	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	53
822	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	53
823	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	53
824	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	53
825	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	53
826	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	53
827	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	52
828	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	52
829	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
830	9078	9130	9183	9235	9287	9340	9392	9444	9496	9549	52
831	* 9601	9653	9706	9758	9810	9862	9914	9967	* 019	0071	52
832	92 0123	0176	0228	0280	0332	0384	0436	0489	0541	0593	52
833	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	52
834	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	52
836	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	52
837	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	52
838	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	52
839	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	52
840	4279	4331	4383	4434	4486	4538	4589	4641	4693	4744	52
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	52
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	52
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
844	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	51
845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	51
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
847	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	51
848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	51
849	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	51
850	9419	9470	9521	9572	9623	9674	9725	9776	9827	9879	51
851	* 9930	9981	* 032	0083	0134	0185	0236	0287	0338	0389	51
852	93 0440	0491	0542	0592	0643	0694	0745	0796	0847	0898	51
853	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	51
854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	51
855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	51
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	51
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	51
858	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	51
859	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	51
860	4498	4549	4599	4650	4700	4751	4801	4852	4902	4953	50
861	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	50
862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	50
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	50
864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	50
865	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468	50
866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	50
867	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	50
868	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	50
869	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	50
870	9519	9569	9619	9669	9719	9769	9819	9869	9918	9968	50
871	94 0018	0068	0118	0168	0218	0267	0317	0367	0417	0467	50
872	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964	50
873	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	50
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	50
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455	50
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	50
877	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	49
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	49
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	49
N.	0	1	2	3	4	5	6	7	8	9	D.

TABLE I.

LOGARITHMS OF NUMBERS.

15

N.	0	1	2	3	4	5	6	7	8	9	D.
880	94 4483	4532	4581	4631	4680	4729	4779	4828	4877	4927	49
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	49
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	49
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	49
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	49
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385	49
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	49
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
890	9390	9439	9488	9536	9585	9634	9683	9731	9780	9829	49
891	* 9878	9926	9975	* 024	0073	0121	0170	0219	0267	0316	49
892	95 0365	0414	0462	0511	0560	0608	0657	0706	0754	0803	49
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	49
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
895	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	48
896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
897	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	48
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	48
899	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	48
900	4243	4291	4339	4387	4435	4484	4532	4580	4628	4677	48
901	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	48
902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640	48
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	48
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
905	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080	48
906	7128	7175	7224	7272	7320	7368	7416	7464	7512	7559	48
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
910	9041	9089	9137	9185	9232	9280	9328	9375	9423	9471	48
911	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	48
912	* 9995	* 042	0990	0138	0185	0233	0280	0328	0376	0423	48
913	96 0471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
914	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	47
915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	47
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	47
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	47
920	3788	3835	3882	3929	3977	4024	4071	4118	4165	4212	47
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	47
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	47
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
930	8483	8530	8576	8623	8670	8716	8763	8810	8856	8903	47
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	47
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	47
933	* 9882	9928	9975	* 021	0068	0114	0161	0207	0254	0300	47
934	97 0347	0393	0440	0486	0533	0579	0626	0672	0719	0765	46
935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	46
N.	0	1	2	3	4	5	6	7	8	9	D.

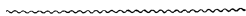
N.	0	1	2	3	4	5	6	7	8	9	D.
940	97 3128	3174	3220	3266	3313	3359	3405	3451	3497	3543	46
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	46
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
950	7724	7769	7815	7861	7906	7952	7998	8043	8089	8135	46
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
955	98 0003	0049	0094	0140	0185	0231	0276	0322	0367	0412	45
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
960	2271	2316	2362	2407	2452	2497	2543	2588	2633	2678	45
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
967	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
970	6772	6817	6861	6906	6951	6996	7040	7085	7130	7175	45
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
975	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	45
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
977	* 9895	9939	9983	+028	0072	0117	0161	0206	0250	0294	44
978	99 0339	0383	0428	0472	0516	0561	0605	0650	0694	0738	44
979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
980	1226	1270	1315	1359	1403	1448	1492	1536	1580	1625	44
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
988	4757	4801	4845	4889	4933	4977	5021	5065	5109	5152	44
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	5635	5679	5723	5767	5811	5854	5898	5942	5986	6030	44
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43
N.	0	1	2	3	4	5	6	7	8	9	D.

T A B L E II.

LOGARITHMIC SINES AND TANGENTS,

FOR

EVERY DEGREE AND MINUTE OF THE QUADRANT.



If the logarithms of the values in Table III. be each increased by 10, the results will be the values of this table.

The logarithmic Secants and Cosecants are not given. They may be readily obtained, as follows:—Subtract the logarithmic Cosine from 20, and the remainder will be the logarithmic Secant; subtract the logarithmic Sine from 20, and the remainder will be the logarithmic Cosecant.

18		LOGARITHMIC SINES, TANGENTS, ETC.						TABLE II.	
0°		179°							
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	Inf. Neg.		10.000000		Inf. Neg.		Infinite.	60	
1	6.463726	501717	000000	00	6.463726	501717	13.536274	59	
2	764756	293485	000000	00	764756	293483	235244	58	
3	940847	208231	000000	00	940847	208231	059153	57	
4	7.065786	161517	000000	00	7.065786	161517	12.934214	56	
5	162696	131968	000000	00	162696	131969	837304	55	
6	241877	111575	9.999999	01	241878	111578	758122	54	
7	368824	96653	999999	01	368825	96653	691175	53	
8	366816	85254	999999	01	366817	85254	633183	52	
9	417968	76263	999999	01	417970	76263	582030	51	
10	463726	68988	999998	01	463727	68988	536273	50	
11	7.505118	62981	9.999998	01	7.505120	62981	12.494880	49	
12	542906	57936	999997	01	542909	57933	457991	48	
13	577668	53641	999997	01	577672	53642	422328	47	
14	609853	49938	999996	01	609857	49939	390143	46	
15	639816	46714	999996	01	639820	46715	360180	45	
16	667845	43881	999995	01	667849	43882	332151	44	
17	694173	41372	999995	01	694179	41373	305821	43	
18	718997	39135	999994	01	719003	39136	280997	42	
19	742478	37127	999993	01	742484	37128	257516	41	
20	764754	35315	999993	01	764761	35136	235239	40	
21	7.785943	33672	9.999992	01	7.785951	33673	12.244049	39	
22	806146	32175	999991	01	806155	32176	193845	38	
23	825451	30805	999990	01	825460	30806	174540	37	
24	843934	29547	999989	02	843944	29549	156056	36	
25	861662	28388	999989	02	861674	28390	138326	35	
26	878695	27317	999988	02	878708	27318	121292	34	
27	895085	26323	999987	02	895099	26325	104901	33	
28	910879	25399	999986	02	910894	25401	089106	32	
29	926119	24538	999985	02	926134	24540	073866	31	
30	940842	23733	999983	02	940858	23735	059142	30	
31	7.955082	22980	9.999982	02	7.955100	22981	12.044900	29	
32	968870	22273	999981	02	968889	22275	031111	28	
33	982233	21608	999980	02	982253	21610	017747	27	
34	995198	20981	999979	02	995219	20983	004781	26	
35	8.007787	20390	999977	02	8.007809	20392	11.992191	25	
36	020021	19831	999976	02	020044	19833	979956	24	
37	031919	19302	999975	02	031945	19305	968055	23	
38	043501	18801	999973	02	043527	18803	956473	22	
39	054781	18325	999972	02	054809	18327	945191	21	
40	065776	17872	999971	02	065806	17874	934194	20	
41	8.076500	17441	9.999969	02	8.076531	17444	11.923469	19	
42	086695	17031	999968	02	086697	17034	913003	18	
43	097183	16639	999966	02	097217	16642	902783	17	
44	107167	16265	999964	03	107203	16268	892797	16	
45	116926	15908	999963	03	116963	15910	883037	15	
46	126471	15566	999961	03	126510	15568	873490	14	
47	135810	15238	999959	03	135851	15241	864149	13	
48	144953	14924	999958	03	144996	14927	855004	12	
49	153907	14622	999956	03	153952	14627	846048	11	
50	162681	14333	999954	03	162727	14336	837273	10	
51	8.171280	14054	9.999952	03	8.171328	14057	11.828672	9	
52	179713	13786	999950	03	179763	13790	820237	8	
53	187985	13529	999948	03	188036	13532	811964	7	
54	196102	13280	999946	03	196156	13284	803844	6	
55	204070	13041	999944	03	204126	13044	795874	5	
56	211895	12810	999942	04	211953	12814	788047	4	
57	219581	12587	999940	04	219641	12590	780359	3	
58	227134	12372	999938	04	227195	12376	772805	2	
59	234557	12164	999936	04	234621	12168	765379	1	
60	241855	11963	999934	04	241921	11967	758079	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	
90°		89°							

TABLE II.

LOGARITHMIC SINES, TANGENTS, ETC.

19

1°								178°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	8.241855	11963	9.999934	04	8.241921	11967	11.758079	60	
1	249033	11768	999932	04	249102	11772	750898	59	
2	256094	11580	999929	04	256165	11584	743835	58	
3	263042	11398	999927	04	263115	11402	736885	57	
4	269881	11221	999925	04	269956	11225	730044	56	
5	276614	11050	999922	04	276691	11054	723309	55	
6	283243	10883	999920	04	283323	10887	716677	54	
7	289773	10721	999918	04	289856	10726	710144	53	
8	296207	10565	999915	04	296292	10570	703708	52	
9	302546	10413	999913	04	302634	10418	697366	51	
10	308794	10266	999910	04	308884	10270	691116	50	
11	8.314954	10122	9.999907	04	8.315046	10126	11.684954	49	
12	321027	9982	999905	04	321122	9987	678878	48	
13	327016	9847	999902	04	327114	9851	672886	47	
14	332924	9714	999899	05	333025	9719	666975	46	
15	338753	9586	999897	05	338856	9590	661144	45	
16	344504	9460	999894	05	344610	9465	655390	44	
17	350181	9338	999891	05	350289	9343	649711	43	
18	355783	9219	999888	05	355895	9224	644105	42	
19	361315	9103	999885	05	361430	9108	638570	41	
20	366777	8990	999882	05	366895	8995	633105	40	
21	8.372171	8880	9.999879	05	8.372292	8885	11.627708	39	
22	377499	8772	999876	05	377622	8777	622378	38	
23	382762	8667	999873	05	382889	8672	617111	37	
24	387962	8564	999870	05	388092	8570	611908	36	
25	393101	8464	999867	05	393234	8470	606766	35	
26	398179	8366	999864	05	398315	8371	601685	34	
27	403199	8271	999861	05	403338	8276	596662	33	
28	408161	8177	999858	05	408304	8182	591696	32	
29	413068	8086	999854	05	413213	8091	586787	31	
30	417919	7996	999851	06	418068	8002	581932	30	
31	8.422717	7909	9.999848	06	8.422869	7914	11.577131	29	
32	427462	7823	999844	06	427618	7830	572382	28	
33	432156	7740	999841	06	432315	7745	567685	27	
34	436800	7657	999838	06	436962	7663	563038	26	
35	441394	7577	999834	06	441560	7583	558440	25	
36	445941	7499	999831	06	446110	7505	553890	24	
37	450440	7422	999827	06	450613	7428	549387	23	
38	454893	7346	999824	06	450700	7352	544930	22	
39	459301	7273	999820	06	450781	7279	540519	21	
40	463665	7200	999816	06	460849	7206	536151	20	
41	8.467985	7129	9.999813	06	8.468172	7135	11.531828	19	
42	472263	7060	999809	06	472454	7066	527546	18	
43	476498	6991	999805	06	476693	6998	523307	17	
44	480693	6924	999801	06	480892	6931	519108	16	
45	484848	6859	999797	07	485050	6865	514950	15	
46	488963	6794	999794	07	489170	6801	510830	14	
47	493040	6731	999790	07	493250	6738	506750	13	
48	497078	6669	999786	07	497293	6676	502707	12	
49	501080	6608	999782	07	501298	6615	498702	11	
50	505045	6548	999778	07	505267	6555	494733	10	
51	8.508974	6489	9.999774	07	8.509200	6496	11.499800	9	
52	512867	6431	999769	07	513098	6439	486902	8	
53	516726	6375	999765	07	516901	6382	483039	7	
54	520551	6319	999761	07	520790	6326	479210	6	
55	524343	6264	999757	07	524586	6272	475414	5	
56	528102	6211	999753	07	528349	6218	471651	4	
57	531828	6158	999748	07	532080	6165	467920	3	
58	535523	6106	999744	07	535779	6113	464221	2	
59	539186	6055	999740	07	539447	6062	460553	1	
60	542819	6004	999735	07	543084	6012	456916	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	
91°								88°	

LOGARITHMIC SINES, TANGENTS, ETC.					TABLE II.				
2°					177°				
'	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	'	
0	8.542819	6004	9.999735	07	8.543084	6012	11.456916	60	
1	546422	5955	999731	07	546691	5962	453309	59	
2	549995	5906	999726	07	550268	5914	449732	58	
3	553539	5858	999722	08	553817	5866	446183	57	
4	557054	5811	999717	08	557336	5819	442664	56	
5	560540	5765	999713	08	560828	5773	439172	55	
6	563999	5719	999708	08	564291	5727	435709	54	
7	567431	5674	999704	08	567727	5682	432273	53	
8	570836	5630	999699	08	571137	5638	428863	52	
9	574214	5587	999694	08	574520	5595	425480	51	
10	577566	5544	999689	08	577877	5552	422123	50	
11	8.580892	5502	9.999685	08	8.581208	5510	11.418792	49	
12	584193	5460	999680	08	584514	5468	415486	48	
13	587469	5419	999675	08	587795	5427	412205	47	
14	590721	5379	999670	08	591051	5387	408949	46	
15	593948	5339	999665	08	594283	5347	405717	45	
16	597152	5300	999660	08	597492	5308	402508	44	
17	600332	5261	999655	08	600677	5270	399323	43	
18	603489	5223	999650	08	603839	5232	396161	42	
19	606623	5186	999645	09	606978	5194	393022	41	
20	609734	5149	999640	09	610094	5158	389906	40	
21	8.612823	5112	9.999635	09	8.613189	5121	11.386811	39	
22	615891	5076	999629	09	616262	5085	383738	38	
23	618937	5041	999624	09	619313	5050	380687	37	
24	621992	5006	999619	09	622343	5015	377657	36	
25	624965	4972	999614	09	625352	4981	374648	35	
26	627948	4938	999608	09	628340	4947	371660	34	
27	630911	4904	999603	09	631308	4913	368692	33	
28	633854	4871	999597	09	634256	4880	365744	32	
29	636776	4839	999592	09	637184	4848	362816	31	
30	639680	4806	999586	09	640093	4816	359907	30	
31	8.642563	4775	9.999581	09	8.642982	4784	11.357018	29	
32	645428	4743	999575	09	645853	4753	354147	28	
33	648274	4712	999570	09	648704	4722	351296	27	
34	651102	4682	999564	09	651537	4691	348463	26	
35	653911	4652	999558	10	654352	4661	345648	25	
36	656702	4622	999553	10	657149	4631	342851	24	
37	659475	4592	999547	10	659928	4602	340072	23	
38	662230	4563	999541	10	662689	4573	337311	22	
39	664968	4535	999535	10	665433	4544	334567	21	
40	667689	4506	999529	10	668160	4526	331840	20	
41	8.670393	4479	9.999524	10	8.670870	4488	11.329130	19	
42	673080	4451	999518	10	673563	4461	326437	18	
43	675751	4424	999512	10	676239	4434	323761	17	
44	678405	4397	999506	10	678900	4417	321100	16	
45	681043	4370	999500	10	681544	4380	318456	15	
46	683665	4344	999493	10	684172	4354	315828	14	
47	686272	4318	999487	10	686784	4328	313216	13	
48	688863	4292	999481	10	689381	4303	310619	12	
49	691438	4267	999475	10	691963	4277	308037	11	
50	693998	4242	999469	10	694529	4252	305471	10	
51	8.696543	4217	9.999463	11	8.697081	4228	11.302919	9	
52	699973	4192	999456	11	699617	4203	300383	8	
53	701589	4168	999450	11	702139	4179	297861	7	
54	704090	4144	999443	11	704646	4155	295354	6	
55	706577	4121	999437	11	707140	4132	292860	5	
56	709049	4097	999431	11	709618	4108	290382	4	
57	711507	4074	999424	11	712083	4085	287917	3	
58	713952	4051	999418	11	714534	4062	285466	2	
59	716383	4029	999411	11	716972	4040	283028	1	
60	718800	4006	999404	11	719396	4017	280604	0	
'	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	'	
92°					87°				

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.								21
3°								176°
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/
0	8.718800	4006	9.999404	11	8.719396	4017	11.280604	60
1	721204	3984	999398	11	721806	3995	278194	59
2	723595	3962	999391	11	724204	3974	275796	58
3	725972	3941	999384	11	726588	3952	273412	57
4	728337	3919	999378	11	728959	3930	271041	56
5	730688	3898	999371	11	731317	3909	268683	55
6	733027	3877	999364	12	733663	3889	266337	54
7	735354	3857	999357	12	735996	3868	264004	53
8	737667	3836	999350	12	738317	3848	261683	52
9	739969	3816	999343	12	740626	3827	259374	51
10	742259	3796	999336	12	742922	3807	257078	50
11	8.744536	3776	9.999329	12	8.745207	3787	11.254793	49
12	746802	3756	999322	12	747479	3768	252521	48
13	749055	3737	999315	12	749740	3749	250260	47
14	751297	3717	999308	12	751989	3729	248011	46
15	753528	3698	999301	12	754227	3710	245773	45
16	755747	3679	999294	12	756453	3692	243547	44
17	757955	3661	999287	12	758668	3673	241332	43
18	760151	3642	999279	12	760872	3655	239128	42
19	762337	3624	999272	12	763065	3636	236935	41
20	764511	3606	999265	12	765246	3618	234754	40
21	8.766675	3588	9.999257	12	8.767417	3600	11.232583	39
22	768802	3570	999250	13	769578	3583	232422	38
23	770970	3553	999242	13	771727	3565	228273	37
24	773101	3535	999235	13	773866	3548	226134	36
25	775223	3518	999227	13	775995	3531	224005	35
26	777333	3501	999220	13	778114	3514	221886	34
27	779434	3484	999212	13	780222	3497	219778	33
28	781524	3467	999205	13	782320	3480	217680	32
29	783605	3451	999197	13	784408	3464	215592	31
30	785675	3431	999189	13	786486	3447	213514	30
31	8.787736	3418	9.999181	13	8.788554	3431	11.211446	29
32	790787	3402	999174	13	790613	3414	209387	28
33	791828	3386	999166	13	792662	3399	207338	27
34	793859	3370	999158	13	794701	3383	205299	26
35	795881	3354	999150	13	796731	3368	203269	25
36	797894	3339	999142	13	798752	3352	201248	24
37	799897	3323	999134	13	800763	3337	199237	23
38	801892	3308	999126	13	802765	3322	197235	22
39	803876	3293	999118	13	804758	3307	195242	21
40	805852	3278	999110	13	806742	3292	193258	20
41	8.807819	3263	9.999102	13	8.808717	3278	11.191283	19
42	809777	3249	999094	14	810683	3262	189317	18
43	811726	3234	999086	14	812641	3248	187359	17
44	813667	3219	999077	14	814589	3233	185411	16
45	815599	3205	999069	14	816529	3219	183471	15
46	817522	3191	999061	14	818461	3205	181539	14
47	819436	3177	999053	14	820384	3191	179616	13
48	821343	3163	999044	14	822298	3177	177702	12
49	823240	3149	999036	14	824205	3163	175795	11
50	825130	3135	999027	14	826103	3150	173897	10
51	8.827011	3122	9.999019	14	8.827992	3136	11.172008	9
52	828884	3108	999010	14	829874	3123	170126	8
53	830749	3095	999002	14	831748	3110	168252	7
54	832607	3082	998993	14	833613	3096	166387	6
55	834456	3069	998984	14	835471	3083	164529	5
56	836297	3056	998976	14	837321	3070	162679	4
57	838130	3043	998967	15	839163	3057	160837	3
58	839956	3030	998958	15	840998	3045	159002	2
59	841774	3017	998950	15	842825	3032	157175	1
60	843585	3000	998941	15	844644	3019	155356	0
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/
93°								86°

22		LOGARITHMIC SINES, TANGENTS, ETC.						TABLE II.	
4°		175°							
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	8.843585	3005	9.998941	15	8.844644	3019	11.155356	60	
1	845387	2992	998932	15	846455	3007	153545	59	
2	847183	2980	998923	15	848260	2995	151740	58	
3	848971	2967	998914	15	850057	2982	149943	57	
4	850751	2955	998905	15	851846	2970	148154	56	
5	852525	2943	998896	15	853628	2958	146372	55	
6	854291	2931	998887	15	855403	2946	144597	54	
7	856049	2919	998878	15	857171	2935	142829	53	
8	857801	2907	998869	15	858932	2923	141068	52	
9	859546	2896	998860	15	860686	2911	139314	51	
10	861283	2884	998851	15	862433	2900	137567	50	
11	8.863014	2873	9.998841	15	8.864173	2888	11.135827	49	
12	864738	2861	998832	15	865906	2877	134094	48	
13	866455	2850	998823	16	867632	2866	132368	47	
14	868165	2839	998813	16	869351	2854	130649	46	
15	869868	2828	998804	16	871064	2843	128936	45	
16	871565	2817	998795	16	872770	2832	127230	44	
17	873255	2806	998785	16	874469	2821	125531	43	
18	874938	2795	998776	16	876162	2811	123838	42	
19	876615	2786	998766	16	877849	2800	122151	41	
20	878285	2773	998757	16	879529	2789	120471	40	
21	8.879949	2763	9.998747	16	8.881202	2779	11.118798	39	
22	881607	2752	998738	16	882869	2768	117131	38	
23	883258	2742	998728	16	884530	2758	115470	37	
24	884903	2731	998718	16	886185	2747	113815	36	
25	886542	2721	998708	16	887833	2737	112167	35	
26	888174	2711	998699	16	889476	2727	110524	34	
27	889801	2700	998689	16	891112	2717	108888	33	
28	891421	2690	998679	16	892742	2707	107258	32	
29	893035	2680	998669	17	894366	2697	105634	31	
30	894643	2670	998659	17	895984	2687	104016	30	
31	8.896246	2660	9.998649	17	8.897596	2677	11.102404	29	
32	897842	2651	998639	17	899203	2667	100797	28	
33	899432	2641	998629	17	900803	2658	999197	27	
34	901017	2631	998619	17	902398	2648	997602	26	
35	902596	2622	998609	17	903987	2638	996013	25	
36	904169	2612	998599	17	905570	2629	994430	24	
37	905736	2603	998589	17	907147	2620	992853	23	
38	907297	2593	998578	17	908719	2610	991281	22	
39	908853	2584	998568	17	910285	2601	989715	21	
40	910404	2575	998558	17	911846	2592	988154	20	
41	8.911949	2566	9.998548	17	8.913401	2583	11.086599	19	
42	913488	2556	998537	17	914951	2574	986504	18	
43	915022	2547	998527	17	916495	2565	985055	17	
44	916550	2538	998516	18	918034	2556	983606	16	
45	918073	2529	998506	18	919568	2547	982157	15	
46	919591	2520	998495	18	921096	2538	980708	14	
47	921103	2512	998485	18	922619	2530	979259	13	
48	922610	2503	998474	18	924136	2521	977810	12	
49	924112	2494	998464	18	925649	2512	976351	11	
50	925609	2486	998453	18	927156	2503	974894	10	
51	8.927100	2477	9.998442	18	8.928658	2495	11.071342	9	
52	928587	2469	998431	18	930155	2486	969845	8	
53	930068	2460	998421	18	931647	2478	968353	7	
54	931544	2452	998410	18	933134	2470	966866	6	
55	933015	2443	998399	18	934616	2461	965384	5	
56	934481	2435	998388	18	936093	2453	963907	4	
57	935942	2427	998377	18	937565	2445	962435	3	
58	937398	2419	998366	18	939032	2437	960968	2	
59	938850	2411	998355	18	940494	2430	959506	1	
60	940296	2403	998344	18	941952	2421	958048	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	
94°		85°							

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.

23

5°								174°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	8.940296	2403	9.998344	19	8.941952	2421	11.058048	60	
1	941738	2394	998333	19	943404	2413	056596	59	
2	943174	2387	998322	19	944852	2405	055148	58	
3	944606	2379	998311	19	946295	2397	053705	57	
4	946034	2371	998300	19	947734	2390	052266	56	
5	947456	2363	998289	19	949168	2382	050832	55	
6	948874	2355	998277	19	950597	2374	049403	54	
7	950287	2348	998266	19	952021	2366	047979	53	
8	951696	2340	998255	19	953441	2360	046559	52	
9	953100	2332	998243	19	954856	2351	045144	51	
10	954499	2325	998232	19	956267	2344	043733	50	
11	8.955894	2317	9.998220	19	8.957674	2337	11.042326	49	
12	957284	2310	998209	19	959075	2329	040925	48	
13	958670	2302	998197	19	960473	2323	039527	47	
14	960052	2295	998186	19	961866	2314	038134	46	
15	961429	2288	998174	19	963255	2307	036745	45	
16	962801	2280	998163	19	964639	2300	035361	44	
17	964170	2273	998151	19	966019	2293	033981	43	
18	965534	2266	998139	20	967394	2286	032606	42	
19	966893	2259	998128	20	968766	2279	031234	41	
20	968249	2252	998116	20	970133	2271	029867	40	
21	8.969600	2244	9.998104	20	8.971496	2265	11.028504	39	
22	970947	2238	998092	20	972855	2257	027145	38	
23	972289	2231	998080	20	974209	2251	025791	37	
24	973628	2224	998068	20	975560	2244	024440	36	
25	974962	2217	998056	20	976906	2237	023094	35	
26	976293	2210	998044	20	978248	2230	021752	34	
27	977619	2203	998032	20	979586	2223	020414	33	
28	978941	2197	998020	20	980921	2217	019079	32	
29	980259	2190	998008	20	982251	2210	017749	31	
30	981573	2183	997996	20	983577	2204	016423	30	
31	8.982883	2177	9.997984	20	8.984899	2197	11.015101	29	
32	984189	2170	997972	20	986217	2191	013783	28	
33	985491	2163	997959	20	987532	2184	012468	27	
34	986789	2157	997947	20	988842	2178	011158	26	
35	988083	2150	997935	21	990149	2171	009851	25	
36	989374	2144	997922	21	991451	2165	008549	24	
37	990660	2138	997910	21	992750	2158	007250	23	
38	991943	2131	997897	21	994045	2152	005955	22	
39	993222	2125	997885	21	995337	2146	004663	21	
40	994497	2119	997872	21	996624	2140	003376	20	
41	8.995768	2112	9.997860	21	8.997908	2134	11.002092	19	
42	997036	2106	997847	21	999188	2127	000812	18	
43	998299	2100	997835	21	9.000465	2121	10.999535	17	
44	999560	2094	997822	21	001738	2115	998262	16	
45	9.000816	2087	997809	21	003007	2109	996993	15	
46	002069	2082	997797	21	004272	2103	995728	14	
47	003318	2076	997784	21	005534	2097	994466	13	
48	004563	2070	997771	21	006792	2091	993208	12	
49	005805	2064	997758	21	008047	2085	991953	11	
50	007044	2058	997745	21	009298	2080	990702	10	
51	9.008278	2052	9.997732	21	9.010546	2074	10.989454	9	
52	009510	2046	997719	21	011790	2068	988210	8	
53	010737	2040	997706	21	013031	2062	986969	7	
54	011962	2034	997693	22	014268	2056	985732	6	
55	013182	2029	997680	22	015502	2051	984498	5	
56	014400	2023	997667	22	016732	2045	983268	4	
57	015613	2017	997654	22	017959	2040	982041	3	
58	016824	2012	997641	22	019183	2033	980817	2	
59	018031	2006	997628	22	020403	2028	979597	1	
60	019235	2000	997614	22	021620	2023	978380	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	
95°								84°	

6°								173°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	9.019235	2000	9.997614	22	9.021620	2023	10.978380	60	
1	020435	1995	997601	22	022834	2017	977166	59	
2	021632	1989	997588	22	024044	2011	975956	58	
3	022825	1984	997574	22	025251	2006	974749	57	
4	024016	1978	997561	22	026455	2000	973545	56	
5	025203	1973	997547	22	027655	1995	972345	55	
6	026386	1967	997534	23	028852	1990	971148	54	
7	027567	1962	997520	23	030046	1985	969954	53	
8	028744	1957	997507	23	031237	1979	968763	52	
9	029918	1951	997493	23	032425	1974	967575	51	
10	031089	1947	997480	23	033609	1969	966391	50	
11	9.032257	1941	9.997466	23	9.034791	1964	10.965209	49	
12	033421	1936	997452	23	035969	1958	964031	48	
13	034582	1930	997439	23	037144	1953	962856	47	
14	035741	1925	997425	23	038316	1948	961684	46	
15	036896	1920	997411	23	039485	1943	960515	45	
16	038048	1915	997397	23	040651	1938	959349	44	
17	039197	1910	997383	23	041813	1933	958187	43	
18	040342	1905	997369	23	042973	1928	957027	42	
19	041485	1899	997355	23	044130	1923	955870	41	
20	042625	1894	997341	23	045284	1918	954716	40	
21	9.043762	1889	9.997327	24	9.046434	1913	10.953566	39	
22	044895	1884	997313	24	047582	1908	952418	38	
23	046026	1879	997299	24	048727	1903	951273	37	
24	047154	1875	997285	24	049869	1898	950131	36	
25	048279	1870	997271	24	051008	1893	948992	35	
26	049400	1865	997257	24	052144	1889	947856	34	
27	050519	1860	997242	24	053277	1884	946723	33	
28	051635	1855	997228	24	054407	1879	945593	32	
29	052749	1850	997214	24	055535	1874	944465	31	
30	053859	1845	997199	24	056659	1870	943341	30	
31	9.054966	1841	9.997185	24	9.057781	1865	10.942219	29	
32	056071	1836	997170	24	058900	1860	941100	28	
33	057172	1831	997156	24	060016	1855	939984	27	
34	058271	1827	997141	24	061130	1851	938870	26	
35	059367	1822	997127	24	062240	1846	937760	25	
36	060460	1817	997112	24	063348	1842	936652	24	
37	061551	1813	997098	24	064453	1837	935547	23	
38	062639	1808	997083	25	065556	1833	934444	22	
39	063724	1804	997068	25	066655	1828	933345	21	
40	064806	1799	997053	25	067752	1824	932248	20	
41	9.065885	1794	9.997039	25	9.068846	1819	10.931154	19	
42	066962	1790	997024	25	069938	1815	930062	18	
43	068036	1786	997009	25	071027	1810	928973	17	
44	069107	1781	996994	25	072113	1806	927887	16	
45	070176	1777	996979	25	073197	1802	926803	15	
46	071242	1772	996964	25	074278	1797	925722	14	
47	072306	1768	996949	25	075356	1793	924644	13	
48	073366	1763	996934	25	076432	1789	923568	12	
49	074424	1759	996919	25	077505	1784	922495	11	
50	075480	1755	996904	25	078576	1780	921424	10	
51	9.076533	1750	9.996889	25	9.079644	1776	10.920356	9	
52	077583	1746	996874	25	080710	1772	919290	8	
53	078631	1742	996858	25	081773	1767	918227	7	
54	079676	1738	996843	25	082833	1763	917167	6	
55	080719	1733	996828	25	083891	1759	916109	5	
56	081759	1729	996812	26	084947	1755	915053	4	
57	082797	1725	996797	26	086000	1751	914000	3	
58	083832	1721	996782	26	087050	1747	912950	2	
59	084864	1717	996766	26	088098	1743	911902	1	
60	085894	1713	996751	26	089144	1738	910856	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	
96°								83°	

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.								25
7°								172°
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/
0	9.085894	1713	9.996751	26	9.089144	1738	10.910856	60
1	086922	1709	996735	26	090187	1734	909813	59
2	087947	1704	996720	26	091228	1730	908772	58
3	088970	1700	996704	26	092266	1727	907734	57
4	089990	1696	996688	26	093302	1722	906698	56
5	091008	1692	996673	26	094336	1719	905664	55
6	092024	1688	996657	26	095367	1715	904633	54
7	093037	1684	996641	26	096395	1711	903605	53
8	094047	1680	996625	26	097422	1707	902578	52
9	095056	1676	996610	26	098446	1703	901554	51
10	096062	1673	996594	26	099468	1699	900532	50
11	9.097065	1668	9.996578	27	9.100487	1695	10.899513	49
12	098066	1665	996562	27	101504	1691	898496	48
13	099065	1661	996546	27	102519	1687	897481	47
14	100062	1657	996530	27	103532	1684	896468	46
15	101056	1653	996514	27	104542	1680	895458	45
16	102048	1649	996498	27	105550	1676	894450	44
17	103037	1645	996482	27	106556	1672	893444	43
18	104025	1641	996465	27	107559	1669	892441	42
19	105010	1638	996449	27	108560	1665	891440	41
20	105992	1634	996433	27	109559	1661	890441	40
21	9.106973	1630	9.996417	27	9.110556	1658	10.889444	39
22	107951	1627	996400	27	111551	1654	888440	38
23	108927	1623	996384	27	112543	1650	887457	37
24	109901	1619	996368	27	113533	1646	886467	36
25	110873	1616	996351	27	114521	1643	885479	35
26	111842	1612	996335	27	115507	1639	884493	34
27	112809	1608	996318	27	116491	1636	883509	33
28	113774	1605	996302	28	117472	1632	882528	32
29	114737	1601	996285	28	118452	1629	881548	31
30	115698	1597	996269	28	119429	1625	880571	30
31	9.116656	1594	9.996252	28	9.120404	1622	10.879596	29
32	117613	1590	996235	28	121377	1618	878623	28
33	118567	1587	996219	28	122348	1615	877652	27
34	119515	1583	996202	28	123317	1611	876683	26
35	120469	1580	996185	28	124284	1607	875716	25
36	121417	1576	996168	28	125249	1604	874751	24
37	122362	1573	996151	28	126211	1601	873789	23
38	123306	1569	996134	28	127172	1597	872828	22
39	124248	1566	996117	28	128130	1594	871870	21
40	125187	1562	996100	28	129087	1591	870913	20
41	9.126125	1559	9.996083	29	9.130041	1587	10.869959	19
42	127060	1556	996066	29	130994	1584	869006	18
43	127993	1552	996049	29	131944	1581	868056	17
44	128925	1549	996032	29	132893	1577	867107	16
45	129854	1545	996015	29	133839	1574	866161	15
46	130781	1542	995998	29	134784	1571	865216	14
47	131706	1539	995980	29	135726	1567	864274	13
48	132630	1535	995963	29	136667	1564	863333	12
49	133551	1532	995946	29	137605	1561	862395	11
50	134470	1529	995928	29	138542	1558	861458	10
51	9.135387	1525	9.995911	29	9.139476	1555	10.860524	9
52	136303	1522	995894	29	140409	1551	859591	8
53	137216	1519	995876	29	141340	1548	858660	7
54	138128	1516	995859	29	142269	1545	857731	6
55	139037	1512	995841	29	143196	1542	856804	5
56	139944	1509	995823	29	144121	1539	855879	4
57	140850	1506	995806	29	145044	1535	854956	3
58	141754	1503	995788	29	145966	1532	854034	2
59	142655	1500	995771	29	146885	1529	853115	1
60	143555	1496	995753	29	147803	1526	852197	0
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/
97°								82°

26		LOGARITHMIC SINES, TANGENTS, ETC.						TABLE II.	
8°								171°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	9.143555	1496	9.995753	30	9.147803	1526	10.852197	60	
1	144453	1493	995735	30	148718	1523	851282	59	
2	145349	1490	995717	30	149632	1520	850368	58	
3	146243	1487	995699	30	150544	1517	849456	57	
4	147136	1484	995681	30	151454	1514	848546	56	
5	148026	1481	995664	30	152363	1511	847637	55	
6	148915	1478	995646	30	153269	1508	846731	54	
7	149802	1475	995628	30	154174	1505	845826	53	
8	150686	1472	995610	30	155077	1502	844923	52	
9	151569	1469	995591	30	155978	1499	844022	51	
10	152451	1466	995573	30	156877	1496	843123	50	
11	9.153330	1463	9.995555	30	9.157775	1493	10.842225	49	
12	154208	1460	995537	30	158671	1490	841329	48	
13	155083	1457	995519	30	159565	1487	840435	47	
14	155957	1454	995501	31	160457	1484	839543	46	
15	156830	1451	995482	31	161347	1481	838653	45	
16	157700	1448	995464	31	162236	1479	837764	44	
17	158566	1445	995446	31	163123	1476	836877	43	
18	159435	1442	995427	31	164008	1473	835992	42	
19	160301	1439	995409	31	164892	1470	835108	41	
20	161164	1436	995390	31	165774	1467	834226	40	
21	9.162025	1433	9.995372	31	9.166654	1464	10.833346	39	
22	162885	1430	995353	31	167532	1461	832468	38	
23	163743	1427	995334	31	168409	1458	831591	37	
24	164600	1424	995316	31	169284	1455	830716	36	
25	165454	1422	995297	31	170157	1453	829843	35	
26	166307	1419	995278	31	171029	1450	828971	34	
27	167159	1416	995260	31	171899	1447	828101	33	
28	168008	1413	995241	32	172767	1444	827233	32	
29	168856	1410	995222	32	173634	1442	826366	31	
30	169702	1407	995203	32	174499	1439	825501	30	
31	9.170547	1405	9.995184	32	9.175362	1436	10.824638	29	
32	171389	1402	995165	32	176224	1433	823776	28	
33	172230	1399	995146	32	177084	1431	822916	27	
34	173070	1396	995127	32	177942	1428	822058	26	
35	173908	1394	995108	32	178799	1425	821201	25	
36	174744	1391	995089	32	179655	1423	820345	24	
37	175578	1388	995070	32	180508	1420	819492	23	
38	176411	1386	995051	32	181360	1417	818640	22	
39	177242	1383	995032	32	182211	1415	817789	21	
40	178072	1380	995013	32	183059	1412	816941	20	
41	9.178900	1377	9.994993	32	9.183907	1409	10.816093	19	
42	179726	1374	994974	32	184752	1407	815248	18	
43	180551	1372	994955	32	185597	1404	814403	17	
44	181374	1369	994935	32	186439	1402	813561	16	
45	182196	1366	994916	33	187280	1399	812720	15	
46	183016	1364	994896	33	188120	1396	811880	14	
47	183834	1361	994877	33	188958	1393	811042	13	
48	184651	1359	994857	33	189794	1391	810206	12	
49	185466	1356	994838	33	190629	1389	809371	11	
50	186280	1353	994818	33	191462	1386	808538	10	
51	9.187092	1351	9.994798	33	9.192294	1384	10.807706	9	
52	187903	1348	994779	33	193124	1381	806876	8	
53	188712	1346	994759	33	193953	1379	806047	7	
54	189519	1343	994739	33	194780	1376	805220	6	
55	190325	1341	994720	33	195606	1374	804394	5	
56	191130	1338	994700	33	196430	1371	803570	4	
57	191933	1336	994680	33	197253	1369	802747	3	
58	192734	1333	994660	33	198074	1366	801926	2	
59	193534	1330	994640	33	198894	1364	801106	1	
60	194332	1328	994620	33	199713	1361	800287	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	

98°

81°

98°

81°

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.

27

9°								170°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	9.194332	1328	9.994620	33	9.199713	1361	10.800287	60	
1	195129	1326	994600	33	200529	1359	799471	59	
2	195925	1323	994580	33	201345	1356	798655	58	
3	196719	1321	994560	34	202159	1354	797841	57	
4	197511	1318	994540	34	202971	1352	797029	56	
5	198302	1316	994519	34	203782	1349	796218	55	
6	199091	1313	994499	34	204592	1347	795408	54	
7	199879	1311	994479	34	205400	1345	794600	53	
8	200666	1308	994459	34	206207	1342	793793	52	
9	201451	1306	994438	34	207013	1340	792987	51	
10	202234	1304	994418	34	207817	1338	792183	50	
11	9.203017	1301	9.994398	34	9.208619	1335	10.791381	49	
12	203797	1299	994377	34	209420	1333	790580	48	
13	204577	1296	994357	34	210220	1331	789780	47	
14	205354	1294	994336	34	211018	1328	788982	46	
15	206131	1292	994316	34	211815	1326	788185	45	
16	206906	1289	994295	34	212611	1324	787389	44	
17	207679	1287	994274	35	213405	1321	786595	43	
18	208452	1285	994254	35	214198	1319	785802	42	
19	209222	1282	994233	35	214989	1317	785011	41	
20	209992	1280	994212	35	215780	1315	784220	40	
21	9.210760	1278	9.994191	35	9.216568	1312	10.783432	39	
22	211526	1275	994171	35	217356	1310	782644	38	
23	212291	1273	994150	35	218142	1308	781858	37	
24	213055	1271	994129	35	218926	1305	781074	36	
25	213818	1268	994108	35	219710	1303	780290	35	
26	214579	1266	994087	35	220492	1301	779508	34	
27	215338	1264	994066	35	221272	1299	778728	33	
28	216097	1261	994045	35	222052	1297	777948	32	
29	216854	1259	994024	35	222830	1294	777170	31	
30	217609	1257	994003	35	223607	1292	776393	30	
31	9.218363	1255	9.993982	35	9.224382	1290	10.775618	29	
32	219116	1253	993960	35	225156	1288	774844	28	
33	219868	1250	993939	35	225929	1286	774071	27	
34	220618	1248	993918	35	226700	1284	773300	26	
35	221367	1246	993897	36	227471	1281	772529	25	
36	222115	1244	993875	36	228239	1279	771761	24	
37	222861	1242	993854	36	229007	1277	770993	23	
38	223606	1239	993832	36	229773	1275	770227	22	
39	224349	1237	993811	36	230539	1273	769461	21	
40	225092	1235	993789	36	231302	1271	768698	20	
41	9.225833	1233	9.993768	36	9.232065	1269	10.767935	19	
42	226573	1231	993746	36	232826	1267	767174	18	
43	227311	1228	993725	36	233586	1265	766414	17	
44	228048	1226	993703	36	234345	1262	765655	16	
45	228784	1224	993681	36	235103	1260	764897	15	
46	229518	1222	993660	36	235859	1258	764141	14	
47	230252	1220	993638	36	236614	1256	763386	13	
48	230984	1218	993616	36	237368	1254	762632	12	
49	231715	1216	993594	37	238120	1252	761880	11	
50	232444	1214	993572	37	238872	1250	761128	10	
51	9.233172	1212	9.993550	37	9.239522	1248	10.760378	9	
52	233899	1209	993528	37	240371	1246	759629	8	
53	234625	1207	993506	37	241118	1244	758882	7	
54	235349	1205	993484	37	241865	1242	758135	6	
55	236073	1203	993462	37	242610	1240	757390	5	
56	236795	1201	993440	37	243354	1238	756646	4	
57	237515	1199	993418	37	244097	1236	755903	3	
58	238235	1197	993396	37	244839	1234	755161	2	
59	238953	1195	993374	37	245579	1232	754421	1	
60	239670	1193	993351	37	246319	1230	753681	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	
99°								80°	

28		LOGARITHMIC SINES, TANGENTS, ETC.						TABLE II.	
10°								169°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	9.239670	1193	9.993351	37	9.246319	1230	10.753681	60	
1	240386	1191	993329	37	247057	1228	752943	59	
2	241101	1189	993307	37	247794	1226	752206	58	
3	241814	1187	993284	37	248530	1224	751470	57	
4	242526	1185	993262	37	249264	1222	750736	56	
5	243237	1183	993240	37	249998	1220	750002	55	
6	243947	1181	993217	38	250730	1218	749270	54	
7	244656	1179	993195	38	251461	1217	748539	53	
8	245363	1177	993172	38	252191	1215	747809	52	
9	246069	1175	993149	38	252920	1213	747080	51	
10	246775	1173	993127	38	253648	1211	746352	50	
11	9.247478	1171	9.993104	38	9.254374	1209	10.745626	49	
12	248181	1169	993081	38	255100	1207	744900	48	
13	248883	1167	993059	38	255824	1205	744176	47	
14	249583	1165	993036	38	256547	1203	743453	46	
15	250282	1163	993013	38	257269	1201	742731	45	
16	250980	1161	992990	38	257990	1200	742010	44	
17	251677	1159	992967	38	258710	1198	741290	43	
18	252373	1158	992944	38	259429	1196	740571	42	
19	253067	1156	992921	38	260146	1194	739854	41	
20	253761	1154	992898	38	260863	1192	739137	40	
21	9.254453	1152	9.992875	38	9.261578	1190	10.738422	39	
22	255144	1150	992852	38	262292	1189	737708	38	
23	255834	1148	992829	39	263005	1187	736995	37	
24	256523	1146	992806	39	263717	1185	736283	36	
25	257211	1144	992783	39	264428	1183	735572	35	
26	257898	1142	992759	39	265138	1181	734862	34	
27	258583	1141	992736	39	265847	1179	734153	33	
28	259268	1139	992713	39	266555	1178	733445	32	
29	259951	1137	992690	39	267261	1176	732739	31	
30	260633	1135	992666	39	267967	1174	732033	30	
31	9.261314	1133	9.992643	39	9.268671	1172	10.731329	29	
32	261994	1131	992619	39	269375	1170	730625	28	
33	262673	1130	992596	39	270077	1169	729923	27	
34	263351	1128	992572	39	270779	1167	729221	26	
35	264027	1126	992549	39	271479	1165	728521	25	
36	264703	1124	992525	39	272178	1164	727822	24	
37	265377	1122	992501	39	272876	1162	727124	23	
38	266051	1120	992478	40	273573	1160	726427	22	
39	266723	1119	992454	40	274269	1158	725731	21	
40	267395	1117	992430	40	274964	1157	725036	20	
41	9.268065	1115	9.992406	40	9.275658	1155	10.724342	19	
42	268734	1113	992382	40	276351	1153	723649	18	
43	269402	1111	992359	40	277043	1151	722957	17	
44	270069	1110	992335	40	277734	1150	722266	16	
45	270735	1108	992311	40	278424	1148	721576	15	
46	271400	1106	992287	40	279113	1147	720887	14	
47	272064	1105	992263	40	279801	1145	720199	13	
48	272726	1103	992239	40	280488	1143	719512	12	
49	273388	1101	992214	40	281174	1141	718826	11	
50	274049	1099	992190	40	281858	1140	718142	10	
51	9.274708	1098	9.992166	40	9.282542	1138	10.717458	9	
52	275367	1096	992142	40	283225	1136	716775	8	
53	276025	1094	992118	41	283907	1135	716093	7	
54	276681	1092	992093	41	284588	1133	715412	6	
55	277337	1091	992069	41	285268	1131	714732	5	
56	277991	1089	992044	41	285947	1130	714053	4	
57	278645	1087	992020	41	286624	1128	713376	3	
58	279297	1086	991996	41	287301	1126	712699	2	
59	279948	1084	991971	41	287977	1125	712023	1	
60	280599	1082	991947	41	288652	1123	711348	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	
100°								79°	

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.								29
11°								168°
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/
0	9.280599	1082	9.991947	41	9.288652	1123	10.711348	60
1	281248	1081	991922	41	289326	1122	710674	59
2	281897	1079	991897	41	289999	1120	710001	58
3	282544	1077	991873	41	290671	1118	709329	57
4	283190	1076	991848	41	291342	1117	708658	56
5	283836	1074	991823	41	292013	1115	707987	55
6	284480	1072	991799	41	292682	1114	707318	54
7	285124	1071	991774	42	293350	1112	706650	53
8	285766	1069	991749	42	294017	1111	705983	52
9	286408	1067	991724	42	294684	1109	705316	51
10	287048	1066	991699	42	295349	1107	704651	50
11	9.287688	1064	9.991674	42	9.296013	1106	10.703987	49
12	288326	1063	991649	42	296677	1104	703323	48
13	288964	1061	991624	42	297339	1103	702661	47
14	289600	1059	991599	42	298001	1101	701999	46
15	290236	1058	991574	42	298662	1100	701338	45
16	290870	1056	991549	42	299322	1098	700678	44
17	291504	1054	991524	42	299980	1096	700020	43
18	292137	1053	991498	42	300638	1095	699362	42
19	292768	1051	991473	42	301295	1093	698705	41
20	293399	1050	991448	42	301951	1092	698049	40
21	9.294029	1048	9.991422	42	9.302607	1090	10.697393	39
22	294658	1046	991397	42	303261	1089	696739	38
23	295286	1045	991372	43	303914	1087	696086	37
24	295913	1043	991346	43	304567	1086	695433	36
25	296539	1042	991321	43	305218	1084	694782	35
26	297164	1040	991295	43	305869	1083	694131	34
27	297788	1039	991270	43	306519	1081	693481	33
28	298412	1037	991244	43	307168	1080	692832	32
29	299034	1036	991218	43	307816	1078	692184	31
30	299655	1034	991193	43	308463	1077	691537	30
31	9.300276	1032	9.991167	43	9.309109	1075	10.690891	29
32	300895	1031	991141	43	309754	1074	690246	28
33	301514	1029	991115	43	310399	1073	689601	27
34	302132	1028	991090	43	311042	1071	688958	26
35	302748	1026	991064	43	311685	1070	688315	25
36	303364	1025	991038	43	312327	1068	687673	24
37	303979	1023	991012	43	312968	1067	687032	23
38	304593	1022	990986	43	313608	1065	686392	22
39	305207	1020	990960	43	314247	1064	685753	21
40	305819	1019	990934	44	314885	1062	685115	20
41	9.306430	1017	9.990908	44	9.315523	1061	10.684477	19
42	307041	1016	990882	44	316159	1060	683841	18
43	307650	1014	990855	44	316795	1058	683205	17
44	308259	1013	990829	44	317430	1057	682570	16
45	308867	1011	990803	44	318064	1055	681936	15
46	309474	1010	990777	44	318697	1054	681303	14
47	310080	1008	990750	44	319330	1053	680670	13
48	310685	1007	990724	44	319961	1051	680039	12
49	311289	1005	990697	44	320592	1050	679408	11
50	311893	1004	990671	44	321222	1048	678778	10
51	9.312495	1003	9.990645	44	9.321851	1047	10.678149	9
52	313097	1001	990618	44	322479	1045	677521	8
53	313698	1000	990591	44	323106	1044	676894	7
54	314297	998	990565	44	323733	1043	676267	6
55	314897	997	990538	44	324358	1041	675642	5
56	315495	996	990511	45	324983	1040	675017	4
57	316092	994	990485	45	325607	1039	674393	3
58	316689	993	990458	45	326231	1037	673769	2
59	317284	991	990431	45	326853	1036	673147	1
60	317879	990	990404	45	327475	1035	672525	0
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/
101°								78°

30		LOGARITHMIC SINES, TANGENTS, ETC.						TABLE II.	
12°								167°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	9·317879	990	9·990404	45	9·327475	1035	10·672525	60	
1	318473	988	990378	45	328095	1033	671905	59	
2	319066	987	990351	45	328715	1032	671285	58	
3	319658	986	990324	45	329334	1030	670666	57	
4	320249	984	990297	45	329953	1029	670047	56	
5	320840	983	990270	45	330570	1028	669430	55	
6	321430	982	990243	45	331187	1026	668813	54	
7	322019	980	990215	45	331803	1025	668197	53	
8	322607	979	990188	45	332418	1024	667582	52	
9	323194	977	990161	45	333033	1023	666967	51	
10	323780	976	990134	45	333646	1021	666354	50	
11	9·324366	975	9·990107	46	9·334259	1020	10·665741	49	
12	324950	973	990079	46	334871	1019	665129	48	
13	325534	972	990052	46	335482	1017	664518	47	
14	326117	970	990025	46	336093	1016	663907	46	
15	326700	969	989997	46	336702	1015	663298	45	
16	327281	968	989970	46	337311	1013	662689	44	
17	327862	966	989942	46	337919	1012	662081	43	
18	328442	965	989915	46	338527	1011	661473	42	
19	329021	964	989887	46	339133	1010	660867	41	
20	329599	962	989860	46	339739	1008	660261	40	
21	9·330176	961	9·989832	46	9·340344	1007	10·659656	39	
22	330753	960	989804	46	340948	1006	659052	38	
23	331329	958	989777	46	341552	1004	658448	37	
24	331903	957	989749	47	342155	1003	657845	36	
25	332478	956	989721	47	342757	1002	657243	35	
26	333051	954	989693	47	343358	1000	656642	34	
27	333624	953	989665	47	343958	999	656042	33	
28	334195	952	989637	47	344558	998	655442	32	
29	334767	950	989610	47	345157	997	654843	31	
30	335337	949	989582	47	345755	996	654245	30	
31	9·335906	948	9·989553	47	9·346353	994	10·653647	29	
32	336475	946	989525	47	346949	993	653051	28	
33	337043	945	989497	47	347545	992	652455	27	
34	337610	944	989469	47	348141	991	651859	26	
35	338176	943	989441	47	348735	990	651265	25	
36	338742	941	989413	47	349329	988	650671	24	
37	339307	940	989385	47	349922	987	650078	23	
38	339871	939	989356	47	350514	986	649486	22	
39	340434	937	989328	47	351106	985	648894	21	
40	340996	936	989300	47	351697	983	648303	20	
41	9·341558	935	9·989271	47	9·352287	982	10·647713	19	
42	342119	934	989243	47	352876	981	647124	18	
43	342679	932	989214	47	353465	980	646535	17	
44	343239	931	989186	47	354053	979	645947	16	
45	343797	930	989157	47	354640	977	645360	15	
46	344355	929	989128	48	355227	976	644773	14	
47	344912	927	989100	48	355813	975	644187	13	
48	345469	926	989071	48	356398	974	643602	12	
49	346024	925	989042	48	356982	973	643018	11	
50	346579	924	989014	48	357566	971	642434	10	
51	9·347134	922	9·988985	48	9·358149	970	10·641851	9	
52	347687	921	988956	48	358731	969	641269	8	
53	348240	920	988927	48	359313	968	640687	7	
54	348792	919	988898	48	359893	967	640107	6	
55	349343	917	988869	48	360474	966	639526	5	
56	349893	916	988840	48	361053	965	638947	4	
57	350443	915	988811	49	361632	963	638368	3	
58	350992	914	988782	49	362210	962	637790	2	
59	351540	913	988753	49	362787	961	637213	1	
60	352088	911	988724	49	363364	960	636636	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	
102°									
77°									

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.								31
13°								166°
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/
0	9.352088	911	9.988724	49	9.363364	960	10.636636	60
1	352635	910	988695	49	363940	959	636060	59
2	353181	909	988666	49	364515	958	635485	58
3	353726	908	988636	49	365090	957	634910	57
4	354271	907	988607	49	365664	955	634336	56
5	354815	905	988578	49	366237	954	633763	55
6	355358	904	988548	49	366810	953	633190	54
7	355901	903	988519	49	367382	952	632618	53
8	356443	902	988489	49	367953	951	632047	52
9	356984	901	988460	49	368524	950	631476	51
10	357524	899	988430	49	369094	949	630906	50
11	9.358064	898	9.988401	49	9.369663	948	10.630337	49
12	358603	897	988371	49	370232	946	629768	48
13	359141	896	988342	49	370799	945	629201	47
14	359678	895	988312	50	371367	944	628633	46
15	360215	893	988282	50	371933	943	628067	45
16	360752	892	988252	50	372499	942	627501	44
17	361287	891	988223	50	373064	941	626936	43
18	361822	890	988193	50	373629	940	626371	42
19	362356	889	988163	50	374193	939	625807	41
20	362889	888	988133	50	374756	938	625244	40
21	9.363422	887	9.988103	50	9.375319	937	10.624681	39
22	363954	885	988073	50	375881	935	624119	38
23	364485	884	988043	50	376442	934	623558	37
24	365016	883	988013	50	377003	933	622997	36
25	365546	882	987983	50	377563	932	622437	35
26	366075	881	987953	50	378122	931	621878	34
27	366604	880	987922	50	378681	930	621319	33
28	367131	879	987892	50	379239	929	620761	32
29	367659	877	987862	50	379797	928	620203	31
30	368185	876	987832	51	380354	927	619646	30
31	9.368711	875	9.987801	51	9.380910	926	10.619090	29
32	369236	874	987771	51	381466	925	618534	28
33	369761	873	987740	51	382020	924	617980	27
34	370285	872	987710	51	382575	923	617425	26
35	370808	871	987679	51	383129	922	616871	25
36	371330	870	987649	51	383682	921	616318	24
37	371852	869	987618	51	384234	920	615766	23
38	372373	867	987588	51	384786	919	615214	22
39	372894	866	987557	51	385337	918	614663	21
40	373414	865	987526	51	385888	917	614112	20
41	9.373933	864	9.987496	51	9.386438	915	10.613562	19
42	374452	863	987465	51	386987	914	613013	18
43	374970	862	987434	51	387536	913	612464	17
44	375487	861	987403	52	388084	912	611916	16
45	376003	860	987372	52	388631	911	611369	15
46	376519	859	987341	52	389178	910	610822	14
47	377035	858	987310	52	389724	909	610276	13
48	377549	857	987279	52	390270	908	609730	12
49	378063	856	987248	52	390815	907	609185	11
50	378577	854	987217	52	391360	906	608640	10
51	9.379089	853	9.987186	52	9.391903	905	10.608097	9
52	379601	852	987155	52	392447	904	607553	8
53	380113	851	987124	52	392989	903	607011	7
54	380624	850	987092	52	393531	902	606469	6
55	381134	849	987061	52	394073	901	605927	5
56	381643	848	987030	52	394614	900	605386	4
57	382152	847	986998	52	395154	899	604846	3
58	382661	846	986967	52	395694	898	604306	2
59	383168	845	986936	52	396233	897	603767	1
60	383675	844	986904	52	396771	896	603229	0
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/
103°								76°

32		LOGARITHMIC SINES, TANGENTS, ETC.						TABLE II.		
14°									165°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/		
0	9.383675	844	9.986904	52	9.396771	896	10.603229	60		
1	384182	843	986873	53	397309	896	602691	59		
2	384687	842	986841	53	397846	895	602154	58		
3	385192	841	986809	53	398383	894	601617	57		
4	385697	840	986778	53	398919	893	601081	56		
5	386201	839	986746	53	399455	892	600545	55		
6	386704	838	986714	53	399990	891	600010	54		
7	387207	837	986683	53	400524	890	599476	53		
8	387709	836	986651	53	401058	889	598942	52		
9	388210	835	986619	53	401591	888	598409	51		
10	388711	834	986587	53	402124	887	597876	50		
11	9.389211	833	9.986555	53	9.402656	886	10.597344	49		
12	389711	832	986523	53	403187	885	596813	48		
13	390210	831	986491	53	403718	884	596282	47		
14	390708	830	986459	53	404249	883	595751	46		
15	391206	828	986427	53	404778	882	595222	45		
16	391703	827	986395	53	405308	881	594692	44		
17	392199	826	986363	54	405836	880	594164	43		
18	392695	825	986331	54	406364	879	593636	42		
19	393191	824	986299	54	406892	878	593108	41		
20	393685	823	986266	54	407419	877	592581	40		
21	9.394179	822	9.986234	54	9.407945	876	10.592055	39		
22	394673	821	986202	54	408471	875	591529	38		
23	395166	820	986169	54	408996	874	591004	37		
24	395658	819	986137	54	409521	874	590479	36		
25	396150	818	986104	54	410045	873	589955	35		
26	396641	817	986072	54	410569	872	589431	34		
27	397132	817	986039	54	411092	871	588908	33		
28	397621	816	986007	54	411615	870	588385	32		
29	398111	815	985974	54	412137	869	587863	31		
30	398600	814	985942	54	412658	868	587342	30		
31	9.399088	813	9.985909	55	9.413179	867	10.586821	29		
32	399575	812	985876	55	413699	866	586301	28		
33	400062	811	985843	55	414219	865	585781	27		
34	400549	810	985811	55	414738	864	585262	26		
35	401035	809	985778	55	415257	864	584743	25		
36	401520	808	985745	55	415775	863	584225	24		
37	402005	807	985712	55	416293	862	583707	23		
38	402489	806	985679	55	416810	861	583190	22		
39	402972	805	985646	55	417326	860	582674	21		
40	403455	804	985613	55	417842	859	582158	20		
41	9.403938	803	9.985580	55	9.418358	858	10.581642	19		
42	404420	802	985547	55	418873	857	581127	18		
43	404901	801	985514	55	419387	856	580613	17		
44	405382	800	985480	55	419901	855	580099	16		
45	405862	799	985447	55	420415	855	579585	15		
46	406341	798	985414	56	420927	854	579073	14		
47	406820	797	985381	56	421440	853	578560	13		
48	407299	796	985347	56	421952	852	578048	12		
49	407777	795	985314	56	422463	851	577537	11		
50	408254	794	985280	56	422974	850	577026	10		
51	9.408731	794	9.985247	56	9.423484	849	10.576516	9		
52	409207	793	985213	56	423993	848	576007	8		
53	409682	792	985180	56	424503	848	575497	7		
54	410157	791	985146	56	425011	847	574989	6		
55	410632	790	985113	56	425519	846	574481	5		
56	411106	789	985079	56	426027	845	573973	4		
57	411579	788	985045	56	426534	844	573466	3		
58	412052	787	985011	56	427041	843	572959	2		
59	412524	786	984978	56	427547	843	572453	1		
60	412996	785	984944	56	428052	842	571948	0		
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/		
104°									75°	

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.								33
15°								164°
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/
0	9.412996	785	9.984944	57	9.428052	842	10.571948	60
1	413467	784	984910	57	428558	841	571442	59
2	413938	783	984876	57	429062	840	570938	58
3	414408	783	984842	57	429566	839	570434	57
4	414878	782	984808	57	430070	838	569930	56
5	415347	781	984774	57	430573	838	569427	55
6	415815	780	984740	57	431075	837	568925	54
7	416283	779	984706	57	431577	836	568423	53
8	416751	778	984672	57	432079	835	567921	52
9	417217	777	984638	57	432580	834	567420	51
10	417684	776	984603	57	433080	833	566920	50
11	9.418150	775	9.984560	57	9.433580	832	10.566420	49
12	418615	774	984535	57	434080	832	565920	48
13	419079	773	984500	57	434579	831	565421	47
14	419544	773	984466	57	435078	830	564922	46
15	420007	772	984432	58	435576	829	564424	45
16	420470	771	984397	58	436073	828	563927	44
17	420933	770	984363	58	436570	828	563430	43
18	421395	769	984328	58	437067	827	562933	42
19	421857	768	984294	58	437563	826	562437	41
20	422318	767	984259	58	438059	825	561941	40
21	9.422778	767	9.984224	58	9.438554	824	10.561446	39
22	423238	766	984190	58	439048	823	560952	38
23	423697	765	984155	58	439543	823	560457	37
24	424156	764	984120	58	440036	822	559964	36
25	424615	763	984085	58	440529	821	559471	35
26	425073	762	984050	58	441022	820	558978	34
27	425530	761	984015	58	441514	819	558486	33
28	425987	760	983981	58	442006	819	557994	32
29	426443	760	983946	58	442497	818	557503	31
30	426899	759	983911	58	442988	817	557012	30
31	9.427354	758	9.983875	58	9.443479	816	10.556521	29
32	427809	757	983840	59	443968	816	556032	28
33	428263	756	983805	59	444458	815	555542	27
34	428717	755	983770	59	444947	814	555053	26
35	429170	754	983735	59	445435	813	554565	25
36	429623	753	983700	59	445923	812	554077	24
37	430075	752	983664	59	446411	812	553589	23
38	430527	752	983629	59	446898	811	553102	22
39	430978	751	983594	59	447384	810	552616	21
40	431429	750	983558	59	447870	809	552130	20
41	9.431879	749	9.983523	59	9.448356	809	10.551644	19
42	432329	749	983487	59	448841	808	551159	18
43	432778	748	983452	59	449326	807	550674	17
44	433226	747	983416	59	449810	806	550190	16
45	433675	746	983381	59	450294	806	549706	15
46	434122	745	983345	59	450777	805	549223	14
47	434569	744	983309	59	451260	804	548740	13
48	435016	744	983273	60	451743	803	548257	12
49	435462	743	983238	60	452225	802	547775	11
50	435908	742	983202	60	452706	802	547294	10
51	9.436353	741	9.983166	60	9.453187	801	10.546813	9
52	436798	740	983130	60	453668	800	546332	8
53	437242	740	983094	60	454148	799	545852	7
54	437686	739	983058	60	454628	799	545372	6
55	438129	738	983022	60	455107	798	544893	5
56	438572	737	982986	60	455586	797	544414	4
57	439014	736	982950	60	456064	796	543936	3
58	439456	736	982914	60	456542	796	543458	2
59	439897	735	982878	60	457019	795	542981	1
60	440338	734	982842	60	457496	794	542504	0
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/

105°

74°

16°

163°

/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/
0	9.440338	734	9.982842	60	9.457496	794	10.542504	60
1	440778	733	982805	60	457973	793	542027	59
2	441218	732	982769	61	458449	793	541551	58
3	441658	731	982733	61	458925	792	541075	57
4	442096	731	982696	61	459400	791	540600	56
5	442535	730	982660	61	459875	790	540125	55
6	442973	729	982624	61	460349	790	539651	54
7	443410	728	982587	61	460823	789	539177	53
8	443847	727	982551	61	461297	788	538703	52
9	444284	727	982514	61	461770	788	538230	51
10	444720	726	982477	61	462242	787	537758	50
11	9.445155	725	9.982441	61	9.462715	786	10.537285	49
12	445590	724	982404	61	463186	785	536814	48
13	446025	723	982367	61	463658	785	536342	47
14	446459	723	982331	61	464128	784	535872	46
15	446893	722	982294	61	464599	783	535401	45
16	447326	721	982257	61	465069	783	534931	44
17	447759	720	982220	62	465539	782	534461	43
18	448191	720	982183	62	466008	781	533992	42
19	448623	719	982146	62	466477	780	533523	41
20	449054	718	982109	62	466945	780	533055	40
21	9.449485	717	9.982072	62	9.467413	779	10.532587	39
22	449915	716	982035	62	467880	778	532120	38
23	450345	716	981998	62	468347	778	531653	37
24	450775	715	981961	62	468814	777	531186	36
25	451204	714	981924	62	469280	776	530720	35
26	451632	713	981886	62	469746	775	530254	34
27	452060	713	981849	62	470211	775	529789	33
28	452488	712	981812	62	470676	774	529324	32
29	452915	711	981774	62	471141	773	528859	31
30	453342	710	981737	62	471605	773	528395	30
31	9.453768	710	9.981700	63	9.472069	772	10.527931	29
32	454194	709	981662	63	472532	771	527468	28
33	454619	708	981625	63	472995	771	527005	27
34	455044	707	981587	63	473457	770	526543	26
35	455469	707	981549	63	473919	769	526081	25
36	455893	706	981512	63	474381	769	525619	24
37	456316	705	981474	63	474842	768	525158	23
38	456739	704	981436	63	475303	767	524697	22
39	457162	704	981399	63	475763	767	524237	21
40	457584	703	981361	63	476223	766	523777	20
41	9.458006	702	9.981323	63	9.476683	765	10.523317	19
42	458427	701	981285	63	477142	765	522858	18
43	458848	701	981247	63	477601	764	522399	17
44	459268	700	981209	63	478059	763	521941	16
45	459688	699	981171	63	478517	763	521483	15
46	460108	698	981133	64	478975	762	521025	14
47	460527	698	981095	64	479432	761	520568	13
48	460946	697	981057	64	479889	761	520111	12
49	461364	696	981019	64	480345	760	519655	11
50	461782	695	980981	64	480801	759	519199	10
51	9.462199	695	9.980942	64	9.481257	759	10.518743	9
52	462616	694	980904	64	481712	758	518288	8
53	463032	693	980866	64	482167	757	517833	7
54	463448	693	980827	64	482621	757	517379	6
55	463864	692	980789	64	483075	756	516925	5
56	464279	691	980750	64	483529	755	516471	4
57	464694	690	980712	64	483982	755	516018	3
58	465108	690	980673	64	484435	754	515565	2
59	465522	689	980635	64	484887	753	515113	1
60	465935	688	980596	64	485339	753	514661	0
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/

106°

73°

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.

35

17°								162°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	9.465935	688	9.980596	64	9.485339	755	10.514661	60	
1	466348	688	980558	64	485791	752	514209	59	
2	466761	687	980519	65	486242	751	513758	58	
3	467173	686	980480	65	486693	751	513307	57	
4	467585	685	980442	65	487143	750	512857	56	
5	467996	685	980403	65	487593	749	512407	55	
6	468407	684	980364	65	488043	749	511957	54	
7	468817	683	980325	65	488492	748	511508	53	
8	469227	683	980286	65	488941	747	511059	52	
9	469637	682	980247	65	489390	747	510610	51	
10	470046	681	980208	65	489838	746	510162	50	
11	9.470455	680	9.980169	65	9.490286	746	10.509714	49	
12	470863	680	980130	65	490733	745	509267	48	
13	471271	679	980091	65	491180	744	508820	47	
14	471679	678	980052	65	491627	744	508373	46	
15	472086	678	980012	65	492073	743	507927	45	
16	472492	677	979973	65	492519	743	507481	44	
17	472898	676	979934	66	492965	742	507035	43	
18	473304	676	979895	66	493410	741	506590	42	
19	473710	675	979855	66	493854	740	506146	41	
20	474115	674	979816	66	494299	740	505701	40	
21	9.474519	674	9.979776	66	9.494743	740	10.505257	39	
22	474923	673	979737	66	495186	739	504814	38	
23	475327	672	979697	66	495630	738	504370	37	
24	475730	672	979658	66	496073	737	503927	36	
25	476133	671	979618	66	496515	737	503485	35	
26	476536	670	979579	66	496957	736	503043	34	
27	476938	669	979539	66	497399	736	502601	33	
28	477340	669	979499	66	497841	735	502159	32	
29	477741	668	979459	66	498282	734	501718	31	
30	478142	667	979420	66	498722	734	501278	30	
31	9.478542	667	9.979380	66	9.499163	733	10.500837	29	
32	478942	666	979340	66	499603	733	500397	28	
33	479342	665	979300	67	500042	732	499958	27	
34	479741	665	979260	67	500481	731	499519	26	
35	480140	664	979220	67	500920	731	499080	25	
36	480539	663	979180	67	501359	730	498641	24	
37	480937	663	979140	67	501797	730	498203	23	
38	481334	662	979100	67	502235	729	497765	22	
39	481731	661	979059	67	502672	728	497328	21	
40	482128	661	979019	67	503109	728	496891	20	
41	9.482525	660	9.978979	67	9.503546	727	10.496454	19	
42	482921	659	978939	67	503982	727	496018	18	
43	483316	659	978898	67	504418	726	495582	17	
44	483712	658	978858	67	504854	725	495146	16	
45	484107	657	978817	67	505289	725	494711	15	
46	484501	657	978777	67	505724	724	494276	14	
47	484895	656	978737	67	506159	724	493841	13	
48	485289	655	978696	68	506593	723	493407	12	
49	485682	655	978655	68	507027	722	492973	11	
50	486075	654	978615	68	507460	722	492540	10	
51	9.486467	653	9.978574	68	9.507893	721	10.492107	9	
52	486860	653	978533	68	508326	721	491674	8	
53	487251	652	978493	68	508759	720	491241	7	
54	487643	651	978452	68	509191	719	490809	6	
55	488034	651	978411	68	509622	719	490378	5	
56	488424	650	978370	68	510054	718	489946	4	
57	488814	650	978329	68	510485	718	489515	3	
58	489204	649	978288	68	510916	717	489084	2	
59	489593	648	978247	68	511346	716	488654	1	
60	489982	648	978206	68	511776	716	488224	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	

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72°

18°

161°

/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/
0	9.489982	648	9.978206	68	9.511776	716	10.488224	60
1	490371	648	978165	68	512206	716	487794	59
2	490739	647	978124	68	512635	715	487365	58
3	491147	646	978083	69	513064	714	486936	57
4	491535	646	978042	69	513493	714	486507	56
5	491922	645	978001	69	513921	713	486079	55
6	492308	644	977959	69	514349	713	485651	54
7	492695	644	977918	69	514777	712	485223	53
8	493081	643	977877	69	515204	712	484796	52
9	493466	642	977835	69	515631	711	484369	51
10	493851	642	977794	69	516057	710	483943	50
11	9.494236	641	9.977752	69	9.516484	710	10.483516	49
12	494621	641	977711	69	516910	709	483090	48
13	495005	640	977669	69	517335	709	482665	47
14	495388	639	977628	69	517761	708	482239	46
15	495772	639	977586	69	518186	708	481814	45
16	496154	638	977544	70	518610	707	481390	44
17	496537	637	977503	70	519034	706	480966	43
18	496919	637	977461	70	519458	706	480542	42
19	497301	636	977419	70	519882	705	480118	41
20	497682	636	977377	70	520305	705	479695	40
21	9.498064	635	9.977335	70	9.520728	704	10.479272	39
22	498444	634	977293	70	521151	703	478849	38
23	498825	634	977251	70	521573	703	478427	37
24	499204	633	977209	70	521995	703	478005	36
25	499584	632	977167	70	522417	702	477583	35
26	499963	632	977125	70	522838	702	477162	34
27	500342	631	977083	70	523259	701	476741	33
28	500721	631	977041	70	523680	701	476320	32
29	501099	630	976999	70	524100	700	475900	31
30	501476	629	976957	70	524520	699	475480	30
31	9.501854	629	9.976914	70	9.524940	699	10.475060	29
32	502231	628	976872	71	525359	698	474641	28
33	502607	628	976830	71	525778	698	474222	27
34	502984	627	976787	71	526197	697	473803	26
35	503360	626	976745	71	526615	697	473385	25
36	503735	626	976702	71	527033	696	472967	24
37	504110	625	976660	71	527451	696	472549	23
38	504485	625	976617	71	527868	695	472132	22
39	504860	624	976574	71	528285	695	471715	21
40	505234	623	976532	71	528702	694	471298	20
41	9.505608	623	9.976489	71	9.529119	693	10.470881	19
42	505981	622	976446	71	529535	693	470465	18
43	506354	622	976404	71	529951	693	470049	17
44	506727	621	976361	71	530366	692	469634	16
45	507099	620	976318	71	530781	691	469219	15
46	507471	620	976275	71	531196	691	468804	14
47	507843	619	976232	72	531611	690	468389	13
48	508214	619	976189	72	532025	690	467975	12
49	508585	618	976146	72	532439	689	467561	11
50	508956	618	976103	72	532853	689	467147	10
51	9.509326	617	9.976060	72	9.533266	688	10.466734	9
52	509696	616	976017	72	533679	688	466321	8
53	510065	616	975974	72	534092	687	465908	7
54	510434	615	975930	72	534504	687	465496	6
55	510803	615	975887	72	534916	686	465084	5
56	511172	614	975844	72	535328	686	464672	4
57	511540	613	975800	72	535739	685	464261	3
58	511907	613	975757	72	536150	685	463850	2
59	512275	612	975714	72	536561	684	463439	1
60	512642	612	975670	72	536972	684	463028	0
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/

108°

71°

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.								37
19°								160°
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/
0	9.512642	612	9.975670	73	9.536972	684	10.463028	60
1	513009	611	975627	73	537382	683	462618	59
2	513375	611	975583	73	537792	683	462208	58
3	513741	610	975539	73	538202	682	461798	57
4	514107	609	975496	73	538611	682	461389	56
5	514472	609	975452	73	539020	681	460980	55
6	514837	608	975408	73	539429	681	460571	54
7	515202	608	975365	73	539837	680	460163	53
8	515566	607	975321	73	540245	680	459755	52
9	515930	607	975277	73	540653	679	459347	51
10	516294	606	975233	73	541061	679	458939	50
11	9.516657	605	9.975189	73	9.541468	678	10.458532	49
12	517020	605	975145	73	541875	678	458125	48
13	517382	604	975101	73	542281	677	457719	47
14	517745	604	975057	73	542688	677	457312	46
15	518107	603	975013	73	543094	676	456906	45
16	518468	603	974969	74	543499	676	456501	44
17	518829	602	974925	74	543905	675	456095	43
18	519190	601	974880	74	544310	675	455690	42
19	519551	601	974836	74	544715	674	455285	41
20	519911	600	974792	74	545119	674	454881	40
21	9.520271	600	9.974748	74	9.545524	673	10.454476	39
22	520631	599	974703	74	545928	673	454072	38
23	520990	599	974659	74	546331	672	453666	37
24	521349	598	974614	74	546735	672	453265	36
25	521707	598	974570	74	547138	671	452862	35
26	522066	597	974525	74	547540	671	452460	34
27	522424	596	974481	74	547943	670	452057	33
28	522781	596	974436	74	548345	670	451655	32
29	523138	595	974391	74	548747	669	451253	31
30	523495	595	974347	75	549149	669	450851	30
31	9.523852	594	9.974302	75	9.549550	668	10.450450	29
32	524208	594	974257	75	549951	668	450049	28
33	524564	593	974212	75	550352	667	449648	27
34	524920	593	974167	75	550752	667	449248	26
35	525275	592	974122	75	551153	666	448847	25
36	525630	591	974077	75	551552	666	448448	24
37	525984	591	974032	75	551952	665	448048	23
38	526339	590	973987	75	552351	665	447649	22
39	526693	590	973942	75	552750	665	447250	21
40	527046	589	973897	75	553149	664	446851	20
41	9.527400	589	9.973852	75	9.553548	664	10.446452	19
42	527753	588	973807	75	553946	663	446054	18
43	528105	588	973761	75	554344	663	445656	17
44	528458	587	973716	76	554741	662	445259	16
45	528810	587	973671	76	555139	662	444861	15
46	529161	586	973625	76	555536	661	444464	14
47	529513	586	973580	76	555933	661	444067	13
48	529864	585	973535	76	556329	660	443671	12
49	530215	585	973489	76	556725	660	443275	11
50	530565	584	973444	76	557121	659	442879	10
51	9.530915	584	9.973398	76	9.557517	659	10.442483	9
52	531265	583	973352	76	557913	659	442087	8
53	531614	582	973307	76	558308	658	441692	7
54	531963	582	973261	76	558703	658	441297	6
55	532312	581	973215	76	559097	657	440903	5
56	532661	581	973169	76	559491	657	440509	4
57	533009	580	973124	76	559885	656	440115	3
58	533357	580	973078	76	560279	656	439721	2
59	533704	579	973032	77	560673	655	439327	1
60	534052	578	972986	77	561066	655	438934	0
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/
109°								70°

38		LOGARITHMIC SINES, TANGENTS, ETC.						TABLE II.		
20°									159°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/		
0	9.534052	578	9.972986	77	9.561066	655	10.438934	60		
1	534399	577	972940	77	561459	654	438541	59		
2	534745	577	972894	77	561851	654	438149	58		
3	535092	577	972848	77	562244	653	437756	57		
4	535438	576	972802	77	562636	653	437364	56		
5	535783	576	972755	77	563028	653	436972	55		
6	536129	575	972709	77	563419	652	436581	54		
7	536474	574	972663	77	563811	652	436189	53		
8	536818	574	972617	77	564202	651	435798	52		
9	537163	573	972570	77	564593	651	435407	51		
10	537507	573	972524	77	564983	650	435017	50		
11	9.537851	572	9.972478	77	9.565373	650	10.434627	49		
12	538194	572	972431	78	565763	649	434237	48		
13	538538	571	972385	78	566153	649	433847	47		
14	538880	571	972338	78	566542	649	433458	46		
15	539223	570	972291	78	566932	648	433068	45		
16	539565	570	972245	78	567320	648	432680	44		
17	539907	569	972198	78	567709	647	432291	43		
18	540249	569	972151	78	568098	647	431902	42		
19	540590	568	972105	78	568486	646	431514	41		
20	540931	568	972058	78	568873	646	431127	40		
21	9.541272	567	9.972011	78	9.569261	645	10.426877	39		
22	541613	567	971964	78	569648	645	430352	38		
23	541953	566	971917	78	570035	645	429965	37		
24	542293	566	971870	78	570422	644	429578	36		
25	542632	565	971823	78	570809	644	429191	35		
26	542971	565	971776	78	571195	643	428805	34		
27	543310	564	971729	79	571581	643	428419	33		
28	543649	564	971682	79	571967	642	428033	32		
29	543987	563	971635	79	572352	642	427648	31		
30	544325	563	971588	79	572738	642	427262	30		
31	9.544663	562	9.971540	79	9.573123	641	10.426877	29		
32	545000	562	971493	79	573507	641	426493	28		
33	545338	561	971446	79	573892	640	426108	27		
34	545674	561	971398	79	574276	640	425724	26		
35	546011	560	971351	79	574660	639	425340	25		
36	546347	560	971303	79	575044	639	424956	24		
37	546683	559	971256	79	575427	639	424573	23		
38	547019	559	971208	79	575810	638	424190	22		
39	547354	558	971161	79	576193	638	423807	21		
40	547689	558	971113	79	576576	637	423424	20		
41	9.548024	557	9.971066	80	9.576959	637	10.423041	19		
42	548359	557	971018	80	577341	636	422659	18		
43	548693	556	970970	80	577723	636	422277	17		
44	549027	556	970922	80	578104	636	421896	16		
45	549360	555	970874	80	578486	635	421514	15		
46	549693	555	970827	80	578867	635	421133	14		
47	550026	554	970779	80	579248	634	420752	13		
48	550359	554	970731	80	579629	634	420371	12		
49	550692	553	970683	80	580009	634	419991	11		
50	551024	553	970635	80	580389	633	419611	10		
51	9.551356	552	9.970586	80	9.580769	633	10.419231	9		
52	551687	552	970538	80	581149	632	418851	8		
53	552018	552	970490	80	581528	632	418472	7		
54	552349	551	970442	80	581907	632	418093	6		
55	552680	551	970394	80	582286	631	417714	5		
56	553010	550	970345	81	582665	631	417335	4		
57	553341	550	970297	81	583044	630	416956	3		
58	553670	549	970249	81	583422	630	416578	2		
59	554000	549	970200	81	583800	629	416200	1		
60	554329	548	970152	81	584177	629	415823	0		
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/		
110°									69°	

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.

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21°								158°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	9.554329	548	9.970152	81	9.584177	629	10.415823	60	
1	554658	548	970103	81	584555	629	415445	59	
2	554987	547	970055	81	584932	628	415068	58	
3	555315	547	970006	81	585309	628	414691	57	
4	555643	546	969957	81	585686	627	414314	56	
5	555971	546	969909	81	586062	627	413938	55	
6	556299	545	969860	81	586439	627	413561	54	
7	556626	545	969811	81	586815	626	413185	53	
8	556953	544	969762	81	587190	626	412810	52	
9	557280	544	969714	81	587566	625	412434	51	
10	557606	543	969665	81	587941	625	412059	50	
11	9.557932	543	9.969616	82	9.588316	625	10.411684	49	
12	558258	543	969567	82	588691	624	411309	48	
13	558583	542	969518	82	589066	624	410934	47	
14	558909	542	969469	82	589440	623	410560	46	
15	559234	541	969420	82	589814	623	410186	45	
16	559558	541	969370	82	590188	623	409812	44	
17	559883	540	969321	82	590562	622	409438	43	
18	560207	540	969272	82	590935	622	409065	42	
19	560531	539	969223	82	591308	622	408692	41	
20	560855	539	969173	82	591681	621	408319	40	
21	9.561178	538	9.969124	82	9.592054	621	10.407946	39	
22	561501	538	969075	82	592426	620	407574	38	
23	561824	537	969025	82	592799	620	407201	37	
24	562146	537	968976	82	593171	619	406829	36	
25	562468	536	968926	83	593542	619	406458	35	
26	562790	536	968877	83	593914	618	406086	34	
27	563112	536	968827	83	594285	618	405715	33	
28	563433	535	968777	83	594656	618	405344	32	
29	563755	535	968728	83	595027	617	404973	31	
30	564075	534	968678	83	595398	617	404602	30	
31	9.564396	534	9.968628	83	9.595768	617	10.404232	29	
32	564716	533	968578	83	596138	616	403862	28	
33	565036	533	968528	83	596508	616	403492	27	
34	565356	532	968479	83	596878	616	403122	26	
35	565676	532	968429	83	597247	615	402753	25	
36	565995	531	968379	83	597616	615	402384	24	
37	566314	531	968329	83	597985	615	402015	23	
38	566632	531	968278	83	598354	614	401646	22	
39	566951	530	968228	84	598722	614	401278	21	
40	567269	530	968178	84	599091	613	400909	20	
41	9.567587	529	9.968128	84	9.599459	613	10.400541	19	
42	567904	529	968078	84	599827	613	400173	18	
43	568222	528	968027	84	600194	612	399806	17	
44	568539	528	967977	84	600562	612	399438	16	
45	568856	528	967927	84	600929	611	399071	15	
46	569172	527	967876	84	601296	611	398704	14	
47	569488	527	967826	84	601663	611	398337	13	
48	569804	526	967775	84	602029	610	397971	12	
49	570120	526	967725	84	602395	610	397605	11	
50	570435	525	967674	84	602761	610	397239	10	
51	9.570751	525	9.967624	84	9.603127	609	10.396873	9	
52	571066	524	967573	84	603493	609	396507	8	
53	571380	524	967522	85	603858	609	396142	7	
54	571695	523	967471	85	604223	608	395777	6	
55	572009	523	967421	85	604588	608	395412	5	
56	572323	523	967370	85	604953	607	395047	4	
57	572636	522	967319	85	605317	607	394683	3	
58	572950	522	967268	85	605682	607	394318	2	
59	573263	521	967217	85	606046	606	393954	1	
60	573575	521	967166	85	606410	606	393590	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	

111°

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40 LOGARITHMIC SINES, TANGENTS, ETC. TABLE II. 157°								
22°								
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/
0	9.573575	521	9.967166	85	9.606410	606	10.393590	60
1	573888	520	967115	85	606773	606	393227	59
2	574200	520	967064	85	607137	605	392863	58
3	574512	519	967013	85	607500	605	392500	57
4	574824	519	966961	85	607863	604	392137	56
5	575136	519	966910	85	608225	604	391775	55
6	575447	518	966859	85	608588	604	391412	54
7	575758	518	966808	85	608950	603	391050	53
8	576069	517	966756	86	609312	603	390688	52
9	576379	517	966705	86	609674	603	390326	51
10	576689	516	966653	86	610036	602	389964	50
11	9.576999	516	9.966602	86	9.610397	602	10.389603	49
12	577309	516	966550	86	610759	602	389241	48
13	577618	515	966499	86	611120	601	388880	47
14	577927	515	966447	86	611480	601	388520	46
15	578236	514	966395	86	611841	601	388159	45
16	578545	514	966344	86	612201	600	387799	44
17	578853	513	966292	86	612561	600	387439	43
18	579162	513	966240	86	612921	600	387079	42
19	579470	513	966188	86	613281	599	386719	41
20	579777	512	966136	86	613641	599	386359	40
21	9.580085	512	9.966085	87	9.614000	598	10.386000	39
22	580392	511	966033	87	614359	598	385641	38
23	580699	511	965981	87	614718	598	385282	37
24	581005	511	965928	87	615077	597	384923	36
25	581312	510	965876	87	615435	597	384565	35
26	581618	510	965824	87	615793	597	384207	34
27	581924	509	965772	87	616151	596	383849	33
28	582229	509	965720	87	616509	596	383491	32
29	582535	509	965668	87	616867	596	383133	31
30	582840	508	965615	87	617224	595	382776	30
31	9.583145	508	9.965563	87	9.617582	595	10.382418	29
32	583449	507	965511	87	617939	595	382061	28
33	583754	507	965458	87	618295	594	381705	27
34	584058	506	965406	87	618652	594	381348	26
35	584361	506	965353	88	619008	594	380992	25
36	584665	506	965301	88	619364	593	380636	24
37	584968	505	965248	88	619720	593	380280	23
38	585272	505	965195	88	620076	593	379924	22
39	585574	504	965143	88	620432	592	379568	21
40	585877	504	965090	88	620787	592	379213	20
41	9.586179	503	9.965037	88	9.621142	592	10.378858	19
42	586482	503	964984	88	621497	591	378503	18
43	586783	503	964931	88	621852	591	378148	17
44	587085	502	964879	88	622207	590	377793	16
45	587386	502	964826	88	622561	590	377439	15
46	587688	501	964773	88	622915	590	377085	14
47	587989	501	964720	88	623269	589	376731	13
48	588289	501	964666	89	623623	589	376377	12
49	588590	500	964613	89	623976	589	376024	11
50	588890	500	964560	89	624330	588	375670	10
51	9.589190	499	9.964507	89	9.624683	588	10.375317	9
52	589489	499	964454	89	625036	588	374964	8
53	589789	499	964400	89	625388	587	374612	7
54	590088	498	964347	89	625741	587	374259	6
55	590387	498	964294	89	626093	587	373907	5
56	590686	497	964240	89	626445	586	373555	4
57	590984	497	964187	89	626797	586	373203	3
58	591282	497	964133	89	627149	586	372851	2
59	591580	496	964080	89	627501	585	372499	1
60	591878	496	964026	89	627852	585	372148	0
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/
112°								
67°								

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.

41

23°								156°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	9.591878	496	9.964026	89	9.627852	585	10.372148	60	
1	592176	495	963972	89	628203	585	371797	59	
2	592473	495	963929	89	628554	585	371446	58	
3	592770	495	963885	90	628905	584	371095	57	
4	593067	494	963841	90	629255	584	370745	56	
5	593363	494	963797	90	629606	583	370394	55	
6	593659	493	963754	90	629956	583	370044	54	
7	593955	493	963710	90	630306	583	369694	53	
8	594251	493	963666	90	630656	583	369344	52	
9	594547	492	963622	90	631005	582	368995	51	
10	594842	492	963578	90	631355	582	368645	50	
11	9.595137	491	9.963434	90	9.631704	582	10.368296	49	
12	595432	491	963390	90	632053	581	367947	48	
13	595727	491	963345	90	632402	581	367598	47	
14	596021	490	963301	90	632750	581	367250	46	
15	596315	490	963257	90	633099	580	366901	45	
16	596609	489	963213	90	633447	580	366553	44	
17	596903	489	963168	91	633795	580	366205	43	
18	597196	489	963124	91	634143	579	365857	42	
19	597490	488	963079	91	634490	579	365510	41	
20	597783	488	963035	91	634838	579	365162	40	
21	9.598075	487	9.962890	91	9.635185	578	10.364815	39	
22	598368	487	962846	91	635532	578	364468	38	
23	598660	487	962801	91	635879	578	364121	37	
24	598952	486	962757	91	636226	577	363774	36	
25	599244	486	962712	91	636572	577	363428	35	
26	599536	485	962668	91	636919	577	363081	34	
27	599827	485	962623	91	637265	577	362735	33	
28	600118	485	962579	91	637611	576	362389	32	
29	600409	484	962534	91	637956	576	362044	31	
30	600700	484	962490	92	638302	576	361698	30	
31	9.600990	484	9.962343	92	9.638647	575	10.361353	29	
32	601280	483	962298	92	638992	575	361008	28	
33	601570	483	962253	92	639337	575	360663	27	
34	601860	482	962208	92	639682	574	360318	26	
35	602150	482	962163	92	640027	574	359973	25	
36	602439	482	962118	92	640371	574	359629	24	
37	602728	481	962073	92	640716	573	359284	23	
38	603017	481	962028	92	641060	573	358940	22	
39	603305	481	961983	92	641404	573	358596	21	
40	603594	480	961938	92	641747	572	358253	20	
41	9.603882	480	9.961791	92	9.642091	572	10.357909	19	
42	604170	479	961745	92	642434	572	357566	18	
43	604457	479	961700	92	642777	572	357223	17	
44	604745	479	961654	93	643120	571	356880	16	
45	605032	478	961609	93	643463	571	356537	15	
46	605319	478	961563	93	643806	571	356194	14	
47	605606	478	961518	93	644148	570	355852	13	
48	605892	477	961472	93	644490	570	355510	12	
49	606179	477	961426	93	644832	570	355168	11	
50	606465	476	961380	93	645174	569	354826	10	
51	9.606751	476	9.961235	93	9.645516	569	10.354484	9	
52	607036	476	961179	93	645857	569	354143	8	
53	607322	475	961123	93	646199	569	353801	7	
54	607607	475	961067	93	646540	568	353460	6	
55	607892	474	961011	93	646881	568	353119	5	
56	608177	474	960955	93	647222	568	352778	4	
57	608461	474	960899	93	647562	567	352438	3	
58	608745	473	960843	94	647903	567	352097	2	
59	609029	473	960786	94	648243	567	351757	1	
60	609313	473	960730	94	648583	566	351417	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	

113°

66°

42		LOGARITHMIC SINES, TANGENTS, ETC.						TABLE II.	
24°		156°							
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	9.609313	473	9.960730	94	9.648583	566	10.351417	60	
1	609597	472	960674	94	648923	566	351077	59	
2	609880	472	960618	94	649263	566	350737	58	
3	610164	472	960561	94	649602	566	350398	57	
4	610447	471	960505	94	649942	565	350058	56	
5	610729	471	960448	94	650281	565	349719	55	
6	611012	470	960392	94	650620	565	349380	54	
7	611294	470	960335	94	650959	564	349041	53	
8	611576	470	960279	94	651297	564	348703	52	
9	611858	469	960222	94	651636	564	348364	51	
10	612140	469	960165	94	651974	563	348026	50	
11	9.612421	469	9.960109	95	9.652312	563	10.347688	49	
12	612702	468	960052	95	652659	563	347350	48	
13	612983	468	959995	95	652988	563	347012	47	
14	613264	467	959938	95	653326	562	346674	46	
15	613545	467	959882	95	653663	562	346337	45	
16	613825	467	959825	95	654000	562	346000	44	
17	614105	466	959768	95	654337	561	345663	43	
18	614385	466	959711	95	654674	561	345326	42	
19	614665	466	959654	95	655011	561	344989	41	
20	614944	465	959596	95	655348	561	344652	40	
21	9.615223	465	9.959539	95	9.655684	560	10.344316	39	
22	615502	465	959482	95	656020	560	343980	38	
23	615781	464	959425	95	656356	560	343644	37	
24	616060	464	959368	95	656692	559	343308	36	
25	616338	464	959310	96	657028	559	342972	35	
26	616616	463	959253	96	657364	559	342636	34	
27	616894	463	959195	96	657699	559	342301	33	
28	617172	462	959138	96	658034	558	341966	32	
29	617450	462	959080	96	658369	558	341631	31	
30	617727	462	959023	96	658704	558	341296	30	
31	9.618004	461	9.958965	96	9.659039	558	10.340961	29	
32	618281	461	958908	96	659373	557	340627	28	
33	618558	461	958850	96	659708	557	340292	27	
34	618834	460	958792	96	660042	557	339958	26	
35	619110	460	958734	96	660376	557	339624	25	
36	619386	460	958677	96	660710	556	339290	24	
37	619662	459	958619	96	661043	556	338957	23	
38	619938	459	958561	96	661377	556	338623	22	
39	620213	459	958503	97	661710	555	338290	21	
40	620488	458	958445	97	662043	555	337957	20	
41	9.620763	458	9.958387	97	9.662376	555	10.337624	19	
42	621038	457	958329	97	662709	554	337291	18	
43	621313	457	958271	97	663042	554	336958	17	
44	621587	457	958213	97	663375	554	336625	16	
45	621861	456	958154	97	663707	554	336293	15	
46	622135	456	958096	97	664039	553	335961	14	
47	622409	456	958038	97	664371	553	335629	13	
48	622682	455	957979	97	664703	553	335297	12	
49	622956	455	957921	97	665035	553	334965	11	
50	623229	455	957863	97	665366	552	334634	10	
51	9.623502	454	9.957804	97	9.665698	552	10.334302	9	
52	623774	454	957746	98	666029	552	333971	8	
53	624047	454	957687	98	666360	551	333640	7	
54	624319	453	957628	98	666691	551	333309	6	
55	624591	453	957570	98	667021	551	332979	5	
56	624863	453	957511	98	667352	551	332648	4	
57	625135	452	957452	98	667682	550	332318	3	
58	625406	452	957393	98	668013	550	331987	2	
59	625677	452	957335	98	668343	550	331657	1	
60	625948	451	957276	98	668673	550	331327	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	
114°		65°							

114°

65°

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.

43

25°								154°							
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/							
0	9.625948	451	9.957276	98	9.668673	550	10.331327	60							
1	626219	451	957217	98	669002	549	330998	59							
2	626490	451	957158	98	669332	549	330668	58							
3	626760	450	957099	98	669661	549	330339	57							
4	627030	450	957040	98	669991	548	330009	56							
5	627300	450	956981	98	670320	548	329680	55							
6	627570	449	956921	99	670649	548	329351	54							
7	627840	449	956862	99	670977	548	329023	53							
8	628109	449	956803	99	671306	547	328694	52							
9	628378	448	956744	99	671635	547	328365	51							
10	628647	448	956684	99	671963	547	328037	50							
11	9.628916	457	9.956625	99	9.672291	547	10.327709	49							
12	629185	447	956566	99	672619	546	327381	48							
13	629453	447	956506	99	672947	546	327053	47							
14	629721	446	956447	99	673274	546	326726	46							
15	629989	446	956387	99	673602	546	326398	45							
16	630257	446	956327	99	673929	545	326071	44							
17	630524	446	956268	99	674257	545	325743	43							
18	630792	445	956208	100	674584	545	325416	42							
19	631059	445	956148	100	674911	544	325089	41							
20	631326	445	956089	100	675237	544	324763	40							
21	9.631593	444	9.956029	100	9.675564	544	10.324436	39							
22	631859	444	955969	100	675890	544	324110	38							
23	632125	444	955909	100	676217	543	323783	37							
24	632392	443	955849	100	676543	543	323457	36							
25	632658	443	955789	100	676869	543	323131	35							
26	632923	443	955729	100	677194	543	322806	34							
27	633189	442	955669	100	677520	542	322480	33							
28	633454	442	955609	100	677846	542	322154	32							
29	633719	442	955548	100	678171	542	321829	31							
30	633984	441	955488	100	678496	542	321504	30							
31	9.634249	441	9.955428	101	9.678821	541	10.321179	29							
32	634514	440	955368	101	679146	541	320854	28							
33	634778	440	955307	101	679471	541	320529	27							
34	635042	440	955247	101	679795	541	320205	26							
35	635306	439	955186	101	680120	540	319880	25							
36	635570	439	955126	101	680444	540	319556	24							
37	635834	439	955065	101	680768	540	319232	23							
38	636097	438	955005	101	681092	540	318908	22							
39	636360	438	954944	101	681416	539	318584	21							
40	636623	438	954883	101	681740	539	318260	20							
41	9.636886	437	9.954823	101	9.682063	539	10.317937	19							
42	637148	437	954762	101	682387	539	317613	18							
43	637411	437	954701	101	682710	538	317290	17							
44	637673	437	954640	101	683033	538	316967	16							
45	637935	436	954579	101	683356	538	316644	15							
46	638197	436	954518	102	683679	538	316321	14							
47	638458	436	954457	102	684001	537	315999	13							
48	638720	435	954396	102	684324	537	315676	12							
49	638981	435	954335	102	684646	537	315354	11							
50	639242	435	954274	102	684968	537	315032	10							
51	9.639503	434	9.954213	102	9.685290	536	10.314710	9							
52	639764	434	954152	102	685612	536	314388	8							
53	640024	434	954090	102	685934	536	314066	7							
54	640284	433	954029	102	686255	536	313745	6							
55	640544	433	953968	102	686577	535	313423	5							
56	640804	433	953906	102	686898	535	313102	4							
57	641064	432	953845	102	687219	535	312781	3							
58	641324	432	953783	102	687540	535	312460	2							
59	641583	432	953722	103	687861	534	312139	1							
60	641842	431	953660	103	688182	534	311818	0							
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/							

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64°

26°

153°

/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/
0	9.641842	431	9.953660	103	9.688182	534	10.311818	60
1	642101	431	953599	103	688502	534	311498	59
2	642360	431	953537	103	688823	534	311177	58
3	642618	430	953475	103	689143	533	310857	57
4	642877	430	953413	103	689463	533	310537	56
5	643135	430	953352	103	689783	533	310217	55
6	643393	430	953290	103	690103	533	309897	54
7	643650	429	953228	103	690423	533	309577	53
8	643908	429	953166	103	690742	532	309258	52
9	644165	429	953104	103	691062	532	308938	51
10	644423	428	953042	103	691381	532	308619	50
11	9.644680	428	9.952980	104	9.691700	531	10.308300	49
12	644936	428	952918	104	692019	531	307981	48
13	645193	427	952855	104	692338	531	307662	47
14	645450	427	952793	104	692656	531	307344	46
15	645706	427	952731	104	692975	531	307025	45
16	645962	426	952669	104	693293	530	306707	44
17	646218	426	952606	104	693612	530	306388	43
18	646474	426	952544	104	693930	530	306070	42
19	646729	425	952481	104	694248	530	305752	41
20	646984	425	952419	104	694566	529	305434	40
21	9.647240	425	9.952356	104	9.694883	529	10.305117	39
22	647494	424	952294	104	695201	529	304799	38
23	647749	424	952231	104	695518	529	304482	37
24	648004	424	952168	105	695836	529	304164	36
25	648258	424	952106	105	696153	528	303847	35
26	648512	423	952043	105	696470	528	303530	34
27	648766	423	951980	105	696787	528	303213	33
28	649020	423	951917	105	697103	528	302897	32
29	649274	422	951854	105	697420	527	302580	31
30	649527	422	951791	105	697736	527	302264	30
31	9.649781	422	9.951728	105	9.698053	527	10.301947	29
32	650034	422	951665	105	698369	527	301631	28
33	650287	421	951602	105	698685	526	301315	27
34	650539	421	951539	105	699001	526	300999	26
35	650792	421	951476	105	699316	526	300684	25
36	651044	420	951412	105	699632	526	300368	24
37	651297	420	951349	106	699947	526	300053	23
38	651549	420	951286	106	700263	525	299737	22
39	651800	419	951222	106	700578	525	299422	21
40	652052	419	951159	106	700893	525	299107	20
41	9.652304	419	9.951096	106	9.701208	524	10.298792	19
42	652555	418	951032	106	701523	524	298477	18
43	652806	418	950968	106	701837	524	298163	17
44	653057	418	950905	106	702152	524	297848	16
45	653308	418	950841	106	702466	524	297534	15
46	653558	417	950778	106	702781	523	297219	14
47	653808	417	950714	106	703095	523	296905	13
48	654059	417	950650	106	703409	523	296591	12
49	654309	416	950586	106	703722	523	296278	11
50	654558	416	950522	107	704036	522	295964	10
51	9.654808	416	9.950458	107	9.704350	522	10.295650	9
52	655058	416	950394	107	704663	522	295337	8
53	655307	415	950330	107	704976	522	295024	7
54	655556	415	950266	107	705290	522	294710	6
55	655805	415	950202	107	705603	521	294397	5
56	656054	414	950138	107	705916	521	294084	4
57	656302	414	950074	107	706228	521	293772	3
58	656551	414	950010	107	706541	521	293459	2
59	656799	413	949945	107	706854	521	293146	1
60	657047	413	949881	107	707166	520	292834	0
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/

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63°

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.

45

27°								152°
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/
0	9.657047	413	9.949881	107	9.707166	520	10.292834	60
1	657295	413	949816	107	707478	520	292522	59
2	657542	412	949752	107	707790	520	292210	58
3	657790	412	949688	108	708102	520	291898	57
4	658037	412	949623	108	708414	519	291586	56
5	658284	412	949558	108	708726	519	291274	55
6	658531	411	949494	108	709037	519	290963	54
7	658778	411	949429	108	709349	519	290651	53
8	659025	411	949364	108	709660	519	290340	52
9	659271	410	949300	108	709971	518	290029	51
10	659517	410	949235	108	710282	518	289718	50
11	9.659763	410	9.949170	108	9.710593	518	10.289407	49
12	660009	409	949105	108	710904	518	289096	48
13	660255	409	949040	108	711215	518	288785	47
14	660501	409	948975	108	711525	517	288475	46
15	660746	409	948910	108	711836	517	288164	45
16	660991	408	948845	108	712146	517	287854	44
17	661236	408	948780	109	712456	517	287544	43
18	661481	408	948715	109	712766	516	287234	42
19	661726	407	948650	109	713076	516	286924	41
20	661970	407	948584	109	713386	516	286614	40
21	9.662214	407	9.948519	109	9.713696	516	10.286304	39
22	662459	407	948514	109	714005	516	286305	38
23	662703	406	948388	109	714314	515	285686	37
24	662946	406	948323	109	714624	515	285376	36
25	663190	406	948257	109	714933	515	285067	35
26	663433	405	948192	109	715242	515	284758	34
27	663677	405	948126	109	715551	514	284449	33
28	663920	405	948060	109	715860	514	284140	32
29	664163	405	947995	110	716168	514	283832	31
30	664406	404	947929	110	716477	514	283523	30
31	9.664648	404	9.947863	110	9.716785	514	10.283215	29
32	664891	404	947797	110	717093	513	282907	28
33	665133	403	947731	110	717401	513	282599	27
34	665375	403	947665	110	717709	513	282291	26
35	665617	403	947600	110	718017	513	281983	25
36	665859	402	947533	110	718325	513	281675	24
37	666100	402	947467	110	718633	512	281367	23
38	666342	402	947401	110	718940	512	281060	22
39	666583	402	947335	110	719248	512	280752	21
40	666824	401	947269	110	719555	512	280445	20
41	9.667065	401	9.947203	110	9.719862	512	10.280138	19
42	667305	401	947136	111	720169	511	279831	18
43	667546	401	947070	111	720476	511	279524	17
44	667786	400	947004	111	720783	511	279217	16
45	668027	400	946937	111	721089	511	278911	15
46	668267	400	946871	111	721396	511	278604	14
47	668506	399	946804	111	721702	510	278298	13
48	668746	399	946738	111	722009	510	277991	12
49	668986	399	946671	111	722315	510	277685	11
50	669225	399	946604	111	722621	510	277379	10
51	9.669464	398	9.946538	111	9.722927	510	10.277073	9
52	669703	398	946471	111	723232	509	276768	8
53	669942	398	946404	111	723538	509	276462	7
54	670181	397	946337	111	723844	509	276156	6
55	670419	397	946270	112	724149	509	275851	5
56	670658	397	946203	112	724454	509	275546	4
57	670896	397	946136	112	724760	508	275240	3
58	671134	396	946069	112	725065	508	274935	2
59	671372	396	946002	112	725370	508	274630	1
60	671609	396	945935	112	725674	508	274326	0
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/

117°

62°

28°								151°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	9.671609	396	9.945935	112	9.725674	508	10.274326	60	
1	671847	395	945868	112	725979	508	274021	59	
2	672084	395	945800	112	726284	507	273716	58	
3	672321	395	945733	112	726588	507	273412	57	
4	672558	395	945666	112	726892	507	273108	56	
5	672795	394	945598	112	727197	507	272803	55	
6	673032	394	945531	112	727501	507	272499	54	
7	673268	394	945464	113	727805	506	272195	53	
8	673505	394	945396	113	728109	506	271891	52	
9	673741	393	945328	113	728412	506	271588	51	
10	673977	393	945261	113	728716	506	271284	50	
11	9.674213	393	9.945193	113	9.729020	506	10.270980	49	
12	674448	392	945125	113	729323	505	270677	48	
13	674684	392	945058	113	729626	505	270374	47	
14	674919	392	944990	113	729929	505	270071	46	
15	675155	392	944922	113	730233	505	269767	45	
16	675390	391	944854	113	730535	505	269465	44	
17	675624	391	944786	113	730838	504	269162	43	
18	675859	391	944718	113	731141	504	268859	42	
19	676094	391	944650	113	731444	504	268556	41	
20	676328	390	944582	114	731746	504	268254	40	
21	9.676562	390	9.944514	114	9.732048	504	10.267952	39	
22	676796	390	944446	114	732351	503	267649	38	
23	677030	390	944377	114	732653	503	267347	37	
24	677264	389	944309	114	732955	503	267045	36	
25	677498	389	944241	114	733257	503	266743	35	
26	677731	389	944172	114	733558	503	266442	34	
27	677964	388	944104	114	733860	502	266140	33	
28	678197	388	944036	114	734162	502	265838	32	
29	678430	388	943967	114	734463	502	265537	31	
30	678663	388	943899	114	734764	502	265236	30	
31	9.678895	387	9.943830	114	9.735066	502	10.264934	29	
32	679128	387	943761	114	735367	502	264633	28	
33	679360	387	943693	115	735668	501	264332	27	
34	679592	387	943624	115	735969	501	264031	26	
35	679824	386	943555	115	736269	501	263731	25	
36	680056	386	943486	115	736570	501	263430	24	
37	680288	386	943417	115	736870	501	263130	23	
38	680519	385	943348	115	737171	500	262829	22	
39	680750	385	943279	115	737471	500	262529	21	
40	680982	385	943210	115	737771	500	262229	20	
41	9.681213	385	9.943141	115	9.738071	500	10.261929	19	
42	681443	384	943072	115	738371	500	261629	18	
43	681674	384	943003	115	738671	499	261329	17	
44	681905	384	942934	115	738971	499	261029	16	
45	682135	384	942864	115	739271	499	260729	15	
46	682365	383	942795	116	739570	499	260430	14	
47	682595	383	942726	116	739870	499	260130	13	
48	682825	383	942656	116	740169	499	259831	12	
49	683055	383	942587	116	740468	498	259532	11	
50	683284	382	942517	116	740767	498	259233	10	
51	9.683514	382	9.942448	116	9.741066	498	10.258934	9	
52	683743	382	942378	116	741365	498	258635	8	
53	683972	382	942308	116	741664	498	258336	7	
54	684201	381	942239	116	741962	497	258038	6	
55	684430	381	942169	116	742261	497	257739	5	
56	684658	381	942099	116	742559	497	257441	4	
57	684887	380	942029	116	742858	497	257142	3	
58	685115	380	941959	116	743156	497	256844	2	
59	685343	380	941889	117	743454	497	256546	1	
60	685571	380	941819	117	743752	496	256248	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.

47

29°								150°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	9.685571	380	9.941819	117	9.743752	496	10.256248	60	
1	685799	379	941749	117	744050	496	253950	59	
2	686027	379	941679	117	744348	496	255652	58	
3	686254	379	941609	117	744645	496	255355	57	
4	686482	379	941539	117	744943	496	255057	56	
5	686709	378	941469	117	745240	496	254760	55	
6	686936	378	941398	117	745538	495	254462	54	
7	687163	378	941328	117	745835	495	254165	53	
8	687389	378	941258	117	746132	495	253868	52	
9	687616	377	941187	117	746429	495	253571	51	
10	687843	377	941117	117	746726	495	253274	50	
11	9.688069	377	9.941046	118	9.747023	494	10.252977	49	
12	688295	377	940975	118	747319	494	252681	48	
13	688521	376	940905	118	747616	494	252384	47	
14	688747	376	940834	118	747913	494	252087	46	
15	688972	376	940763	118	748209	494	251791	45	
16	689198	376	940693	118	748505	493	251495	44	
17	689423	375	940622	118	748801	493	251199	43	
18	689648	375	940551	118	749097	493	250903	42	
19	689873	375	940480	118	749393	493	250607	41	
20	690098	375	940409	118	749689	493	250311	40	
21	9.690323	374	9.940338	118	9.749985	493	10.250015	39	
22	690548	374	940267	118	750281	492	249719	38	
23	690772	374	940196	118	750576	492	249424	37	
24	690996	374	940125	119	750872	492	249128	36	
25	691220	373	940054	119	751167	492	248833	35	
26	691444	373	939982	119	751462	492	248538	34	
27	691668	373	939911	119	751757	492	248243	33	
28	691892	373	939840	119	752052	491	247948	32	
29	692115	372	939768	119	752347	491	247653	31	
30	692339	372	939697	119	752642	491	247358	30	
31	9.692562	372	9.939625	119	9.752937	491	10.247063	29	
32	692785	371	939554	119	753231	491	246769	28	
33	693008	371	939482	119	753526	491	246474	27	
34	693231	371	939410	119	753820	490	246180	26	
35	693453	371	939339	119	754115	490	245885	25	
36	693676	370	939267	120	754409	490	245591	24	
37	693898	370	939195	120	754703	490	245297	23	
38	694120	370	939123	120	754997	490	245003	22	
39	694342	370	939052	120	755291	490	244709	21	
40	694564	369	938980	120	755585	489	244415	20	
41	9.694786	369	9.938908	120	9.755878	489	10.244122	19	
42	695007	369	938836	120	756172	489	243828	18	
43	695229	369	938763	120	756465	489	243535	17	
44	695450	368	938691	120	756759	489	243241	16	
45	695671	368	938619	120	757052	489	242948	15	
46	695892	368	938547	120	757345	488	242655	14	
47	696113	368	938475	120	757638	488	242362	13	
48	696334	367	938402	121	757931	488	242069	12	
49	696554	367	938330	121	758224	488	241776	11	
50	696775	367	938258	121	758517	488	241483	10	
51	9.696995	367	9.938185	121	9.758810	488	10.241190	9	
52	697215	366	938113	121	759102	487	240898	8	
53	697435	366	938040	121	759395	487	240605	7	
54	697654	366	937967	121	759687	487	240313	6	
55	697874	366	937895	121	759979	487	240021	5	
56	698094	365	937822	121	760272	487	239728	4	
57	698313	365	937749	121	760564	487	239436	3	
58	698532	365	937676	121	760856	486	239144	2	
59	698751	365	937604	121	761148	486	238852	1	
60	698970	364	937531	121	761439	486	238561	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	

119°

60°

48		LOGARITHMIC SINES, TANGENTS, ETC.						TABLE II.	
30°								149°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	9.698070	364	9.937531	121	9.761439	486	10.238561	60	
1	699189	364	937458	122	761731	486	238269	59	
2	699407	364	937385	122	762023	486	237977	58	
3	699626	364	937312	122	762314	486	237686	57	
4	699844	363	937238	122	762606	485	237394	56	
5	700062	363	937165	122	762897	485	237103	55	
6	700280	363	937092	122	763188	485	236812	54	
7	700498	363	937019	122	763479	485	236521	53	
8	700716	363	936946	122	763770	485	236230	52	
9	700933	362	936872	122	764061	485	235939	51	
10	701151	362	936799	122	764352	484	235648	50	
11	9.701368	362	9.936725	122	9.764643	484	10.235357	49	
12	701585	362	936652	123	764933	484	235067	48	
13	701802	361	936578	123	765224	484	234776	47	
14	702019	361	936505	123	765514	484	234486	46	
15	702236	361	936431	123	765805	484	234195	45	
16	702452	361	936357	123	766095	484	233905	44	
17	702669	360	936284	123	766385	483	233615	43	
18	702885	360	936210	123	766675	483	233325	42	
19	703101	360	936136	123	766965	483	233035	41	
20	703317	360	936062	123	767255	483	232745	40	
21	9.703533	359	9.935988	123	9.767545	483	10.232455	39	
22	703749	359	935914	123	767834	483	232166	38	
23	703964	359	935840	123	768124	482	231876	37	
24	704179	359	935766	124	768414	482	231586	36	
25	704395	359	935692	124	768703	482	231297	35	
26	704610	358	935618	124	768992	482	231008	34	
27	704825	358	935543	124	769281	482	230719	33	
28	705040	358	935469	124	769571	482	230429	32	
29	705254	358	935395	124	769860	481	230140	31	
30	705469	357	935320	124	770148	481	229852	30	
31	9.705683	357	9.935246	124	9.770437	481	10.229563	29	
32	705898	357	935171	124	770726	481	229274	28	
33	706112	357	935097	124	771015	481	228985	27	
34	706326	356	935022	124	771303	481	228697	26	
35	706539	356	934948	124	771592	481	228408	25	
36	706753	356	934873	124	771880	480	228120	24	
37	706967	356	934798	125	772168	480	227832	23	
38	707180	355	934723	125	772457	480	227543	22	
39	707393	355	934649	125	772745	480	227255	21	
40	707606	355	934574	125	773033	480	226967	20	
41	9.707819	355	9.934499	125	9.773321	480	10.226679	19	
42	708032	354	934424	125	773608	479	226392	18	
43	708245	354	934349	125	773896	479	226104	17	
44	708458	354	934274	125	774184	479	225816	16	
45	708670	354	934199	125	774471	479	225529	15	
46	708882	353	934123	125	774759	479	225241	14	
47	709094	353	934048	125	775046	479	224954	13	
48	709306	353	933973	125	775333	479	224667	12	
49	709518	353	933898	126	775621	478	224379	11	
50	709730	353	933822	126	775908	478	224092	10	
51	9.709941	352	9.933747	126	9.776195	478	10.223805	9	
52	710153	352	933671	126	776482	478	223518	8	
53	710364	352	933596	126	776768	478	223232	7	
54	710575	352	933520	126	777055	478	222945	6	
55	710786	351	933445	126	777342	478	222658	5	
56	710997	351	933369	126	777628	477	222372	4	
57	711208	351	933293	126	777915	477	222085	3	
58	711419	351	933217	126	778201	477	221799	2	
59	711629	350	933141	126	778488	477	221512	1	
60	711839	350	933066	126	778774	477	221226	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	
120°								59°	

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.								49
81°								148°
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/
0	9.711839	350	9.933066	126	9.778774	477	10.221226	60
1	712050	350	932090	127	779060	477	220940	59
2	712260	350	932914	127	779346	476	220554	58
3	712469	349	932838	127	779632	476	220368	57
4	712679	349	932762	127	779918	476	220082	56
5	712880	349	932685	127	780203	476	219797	55
6	713098	349	932609	127	780489	476	219511	54
7	713308	349	932533	127	780775	476	219225	53
8	713517	348	932457	127	781060	476	218940	52
9	713726	348	932380	127	781346	475	218654	51
10	713935	348	932304	127	781631	475	218369	50
11	9.714144	348	9.932228	127	9.781916	475	10.218084	49
12	714352	347	932151	127	782201	475	217799	48
13	714561	347	932075	128	782486	475	217514	47
14	714769	347	931998	128	782771	475	217229	46
15	714978	347	931921	128	783056	475	216944	45
16	715186	347	931845	128	783341	475	216659	44
17	715394	346	931768	128	783626	474	216374	43
18	715602	346	931691	128	783910	474	216090	42
19	715809	346	931614	128	784195	474	215805	41
20	716017	346	931537	128	784479	474	215521	40
21	9.716224	345	9.931460	128	9.784764	474	10.215236	39
22	716432	345	931383	128	785048	474	214952	38
23	716639	345	931306	128	785332	473	214668	37
24	716846	345	931229	129	785616	473	214384	36
25	717053	345	931152	129	785900	473	214100	35
26	717259	344	931075	129	786184	473	213816	34
27	717466	344	930998	129	786468	473	213532	33
28	717673	344	930921	129	786752	473	213248	32
29	717879	344	930843	129	787036	473	212964	31
30	718085	343	930766	129	787319	472	212681	30
31	9.718291	343	9.930688	129	9.787603	472	10.212397	29
32	718497	343	930611	129	787886	472	212114	28
33	718703	343	930533	129	788170	472	211830	27
34	718909	343	930456	129	788453	472	211547	26
35	719114	342	930378	129	788736	472	211264	25
36	719320	342	930300	130	789019	472	210981	24
37	719525	342	930223	130	789302	471	210698	23
38	719730	342	930145	130	789585	471	210415	22
39	719935	341	930067	130	789868	471	210132	21
40	720140	341	929989	130	790151	471	209849	20
41	9.720345	341	9.929911	130	9.790434	471	10.209566	19
42	720549	341	929833	130	790716	471	209284	18
43	720754	340	929755	130	790999	471	209001	17
44	720958	340	929677	130	791281	471	208719	16
45	721162	340	929599	130	791563	470	208437	15
46	721366	340	929521	130	791846	470	208154	14
47	721570	340	929442	130	792128	470	207872	13
48	721774	339	929364	131	792410	470	207590	12
49	721978	339	929286	131	792692	470	207308	11
50	722181	339	929207	131	792974	470	207026	10
51	9.722385	339	9.929129	131	9.793256	470	10.206744	9
52	722588	339	929050	131	793538	469	206462	8
53	722791	338	928972	131	793819	469	206181	7
54	722994	338	928893	131	794101	469	205899	6
55	723197	338	928815	131	794383	469	205617	5
56	723400	338	928736	131	794664	469	205336	4
57	723603	337	928657	131	794946	469	205054	3
58	723805	337	928578	131	795227	469	204773	2
59	724007	337	928499	131	795508	468	204492	1
60	724210	337	928420	131	795789	468	204211	0
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/
121°								58°

50 LOGARITHMIC SINES, TANGENTS, ETC. TABLE II. 32°								
								147°
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/
0	9.724210	337	9.928420	132	9.795789	468	10.204211	60
1	724412	337	928342	132	79607c	468	203930	59
2	724614	336	928263	132	796351	468	203649	58
3	724816	336	928183	132	796632	468	203368	57
4	725017	336	928104	132	796913	468	203087	56
5	725219	336	928025	132	797194	468	202806	55
6	725420	335	927946	132	797474	468	202526	54
7	725622	335	927867	132	797755	468	202245	53
8	725823	335	927787	132	798036	467	201964	52
9	726024	335	927708	132	798316	467	201684	51
10	726225	335	927629	132	798596	467	201404	50
11	9.726426	334	9.927549	132	9.798877	467	10.201123	49
12	726626	334	927470	133	799157	467	200843	48
13	726827	334	927390	133	799437	467	200563	47
14	727027	334	927310	133	799717	467	200283	46
15	727228	334	927231	133	799997	466	200003	45
16	727428	333	927151	133	800277	466	199723	44
17	727628	333	927071	133	800557	466	199443	43
18	727828	333	926991	133	800836	466	199164	42
19	728027	333	926911	133	801116	466	198884	41
20	728227	333	926831	133	801396	466	198604	40
21	9.728427	332	9.926751	133	9.801675	466	10.198325	39
22	728626	332	926671	133	801955	466	198045	38
23	728825	332	926591	133	802234	465	197766	37
24	729024	332	926511	134	802513	465	197487	36
25	729223	331	926431	134	802792	465	197208	35
26	729422	331	926351	134	803072	465	196929	34
27	729621	331	926270	134	803351	465	196649	33
28	729820	331	926190	134	803630	465	196370	32
29	730018	330	926110	134	803909	465	196091	31
30	730217	330	926029	134	804187	465	195813	30
31	9.730415	330	9.925949	134	9.804466	464	10.195534	29
32	730613	330	925868	134	804745	464	195255	28
33	730811	330	925788	134	805023	464	194977	27
34	731009	329	925707	134	805302	464	194698	26
35	731206	329	925626	134	805580	464	194420	25
36	731404	329	925545	135	805859	464	194141	24
37	731602	329	925465	135	806137	464	193863	23
38	731799	329	925384	135	806415	463	193585	22
39	731996	328	925303	135	806693	463	193307	21
40	732193	328	925222	135	806971	463	193029	20
41	9.732390	328	9.925141	135	9.807249	463	10.192751	19
42	732587	328	925060	135	807527	463	192473	18
43	732784	328	924979	135	807805	463	192195	17
44	732980	327	924897	135	808083	463	191917	16
45	733177	327	924816	135	808361	463	191639	15
46	733373	327	924735	136	808638	462	191362	14
47	733569	327	924654	136	808916	462	191084	13
48	733765	327	924572	136	809193	462	190807	12
49	733961	326	924491	136	809471	462	190529	11
50	734157	326	924409	136	809748	462	190252	10
51	9.734353	326	9.924328	136	9.810025	462	10.189975	9
52	734549	326	924246	136	810302	462	189698	8
53	734744	325	924164	136	810580	462	189420	7
54	734939	325	924083	136	810857	462	189143	6
55	735135	325	924001	136	811134	461	188866	5
56	735330	325	923919	136	811410	461	188590	4
57	735525	325	923837	136	811687	461	188313	3
58	735719	324	923755	137	811964	461	188036	2
59	735914	324	923673	137	812241	461	187759	1
60	736109	324	923591	137	812517	461	187483	0
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/
122°								
								57°

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.

51

33°								146°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	9.736109	324	9.923591	137	9.812517	461	10.187483	60	
1	736303	324	923509	137	812794	461	187206	59	
2	736498	324	923427	137	813070	461	186930	58	
3	736692	323	923345	137	813347	460	186653	57	
4	736886	323	923263	137	813623	460	186377	56	
5	737080	323	923181	137	813899	460	186101	55	
6	737274	323	923098	137	814176	460	185824	54	
7	737467	323	923016	137	814452	460	185548	53	
8	737661	322	922933	137	814728	460	185272	52	
9	737855	322	922851	137	815004	460	184996	51	
10	738048	322	922768	138	815280	460	184720	50	
11	9.738241	322	9.922686	138	9.815555	459	10.184445	49	
12	738434	322	922603	138	815831	459	184169	48	
13	738627	321	922520	138	816107	459	183893	47	
14	738820	321	922438	138	816382	459	183618	46	
15	739013	321	922355	138	816658	459	183342	45	
16	739206	321	922272	138	816933	459	183067	44	
17	739398	321	922189	138	817209	459	182791	43	
18	739590	320	922106	138	817484	459	182516	42	
19	739783	320	922023	138	817759	459	182241	41	
20	739975	320	921940	138	818035	458	181965	40	
21	9.740167	320	9.921857	139	9.818310	458	10.181690	39	
22	740359	320	921774	139	818585	458	181415	38	
23	740550	319	921691	139	818860	458	181140	37	
24	740742	319	921607	139	819135	458	180865	36	
25	740934	319	921524	139	819410	458	180590	35	
26	741125	319	921441	139	819684	458	180316	34	
27	741316	319	921357	139	819959	458	180041	33	
28	741508	318	921274	139	820234	458	179766	32	
29	741699	318	921190	139	820508	457	179492	31	
30	741889	318	921107	139	820783	457	179217	30	
31	9.742080	318	9.921023	139	9.821057	457	10.178943	29	
32	742271	318	920939	140	821332	457	178668	28	
33	742462	317	920856	140	821606	457	178394	27	
34	742652	317	920772	140	821880	457	178120	26	
35	742842	317	920688	140	822154	457	177846	25	
36	743033	317	920604	140	822429	457	177571	24	
37	743223	317	920520	140	822703	457	177297	23	
38	743413	316	920436	140	822977	456	177023	22	
39	743602	316	920352	140	823251	456	176749	21	
40	743792	316	920268	140	823524	456	176476	20	
41	9.743982	316	9.920184	140	9.823798	456	10.176202	19	
42	744171	316	920099	140	824072	456	175928	18	
43	744361	315	920015	140	824345	456	175655	17	
44	744550	315	919931	141	824619	456	175381	16	
45	744739	315	919846	141	824893	456	175107	15	
46	744928	315	919762	141	825166	456	174834	14	
47	745117	315	919677	141	825439	455	174561	13	
48	745306	314	919593	141	825713	455	174287	12	
49	745494	314	919508	141	825986	455	174014	11	
50	745683	314	919424	141	826259	455	173741	10	
51	9.745871	314	9.919339	141	9.826532	455	10.173468	9	
52	746060	314	919254	141	826805	455	173195	8	
53	746248	313	919169	141	827078	455	172922	7	
54	746436	313	919085	141	827351	455	172649	6	
55	746624	313	919000	141	827624	455	172376	5	
56	746812	313	918915	142	827897	454	172103	4	
57	746999	313	918830	142	828170	454	171830	3	
58	747187	312	918745	142	828442	454	171558	2	
59	747374	312	918659	142	828715	454	171285	1	
60	747562	312	918574	142	828987	454	171013	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	

128°

56°

84°

145°

'	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	'
0	9.747562	312	9.918574	142	9.828987	454	10.171013	60
1	747749	312	918489	142	829260	454	170740	59
2	747936	312	918404	142	829532	454	170468	58
3	748123	311	918318	142	829805	454	170195	57
4	748310	311	918233	142	830077	454	169923	56
5	748497	311	918147	142	830349	453	169651	55
6	748683	311	918062	142	830621	453	169379	54
7	748870	311	917976	143	830893	453	169107	53
8	749056	310	917891	143	831165	453	168835	52
9	749243	310	917805	143	831437	453	168563	51
10	749429	310	917719	143	831709	453	168291	50
11	9.749615	310	9.917634	143	9.831981	453	10.168019	49
12	749801	310	917548	143	832253	453	167747	48
13	749987	309	917462	143	832525	453	167475	47
14	750172	309	917376	143	832796	453	167204	46
15	750358	309	917290	143	833068	452	166932	45
16	750543	309	917204	143	833339	452	166661	44
17	750729	309	917118	144	833611	452	166389	43
18	750914	308	917032	144	833882	452	166118	42
19	751099	308	916946	144	834154	452	165846	41
20	751284	308	916859	144	834425	452	165575	40
21	9.751469	308	9.916773	144	9.834696	452	10.165304	39
22	751654	308	916687	144	834967	452	165033	38
23	751839	308	916600	144	835238	452	164762	37
24	752023	307	916514	144	835509	452	164491	36
25	752208	307	916427	144	835780	451	164220	35
26	752392	307	916341	144	836051	451	163949	34
27	752576	307	916254	144	836322	451	163678	33
28	752760	307	916167	145	836593	451	163407	32
29	752944	306	916081	145	836864	451	163136	31
30	753128	306	915994	145	837134	451	162866	30
31	9.753312	306	9.915907	145	9.837405	451	10.162595	29
32	753495	306	915820	145	837675	451	162325	28
33	753679	306	915733	145	837946	451	162054	27
34	753862	305	915646	145	838216	451	161784	26
35	754046	305	915559	145	838487	450	161513	25
36	754229	305	915472	145	838757	450	161243	24
37	754412	305	915385	145	839027	450	160973	23
38	754595	305	915297	145	839297	450	160703	22
39	754778	304	915210	145	839568	450	160432	21
40	754960	304	915123	146	839838	450	160162	20
41	9.755143	304	9.915035	146	9.840108	450	10.159892	19
42	755326	304	914948	146	840378	450	159622	18
43	755508	304	914860	146	840648	450	159352	17
44	755690	304	914773	146	840917	449	159083	16
45	755872	303	914685	146	841187	449	158813	15
46	756054	303	914598	146	841457	449	158543	14
47	756236	303	914510	146	841727	449	158273	13
48	756418	303	914422	146	841996	449	158004	12
49	756600	303	914334	146	842266	449	157734	11
50	756782	302	914246	147	842535	449	157465	10
51	9.756963	302	9.914158	147	9.842805	449	10.157195	9
52	757144	302	914070	147	843074	449	156926	8
53	757326	302	913982	147	843343	449	156657	7
54	757507	302	913894	147	843612	449	156388	6
55	757688	301	913806	147	843882	448	156118	5
56	757869	301	913718	147	844151	448	155849	4
57	758050	301	913630	147	844420	448	155580	3
58	758230	301	913541	147	844689	448	155311	2
59	758411	301	913453	147	844958	448	155042	1
60	758591	301	913365	147	845227	448	154773	0
'	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	'

124°

55°

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.								53
85°								144°
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/
0	9.758591	301	9.913365	147	9.845227	448	10.154773	60
1	758772	300	913276	147	845496	448	154504	59
2	758952	300	913187	148	845764	448	154236	58
3	759132	300	913099	148	846033	448	153967	57
4	759312	300	913010	148	846302	448	153698	56
5	759492	300	912922	148	846570	447	153430	55
6	759672	299	912833	148	846839	447	153161	54
7	759852	299	912744	148	847108	447	152892	53
8	760031	299	912655	148	847376	447	152624	52
9	760211	299	912566	148	847644	447	152356	51
10	760390	299	912477	148	847913	447	152087	50
11	9.760569	298	9.912388	148	9.848181	447	10.151819	49
12	760748	298	912299	149	848449	447	151551	48
13	760927	298	912210	149	848717	447	151283	47
14	761106	298	912121	149	848986	447	151014	46
15	761285	298	912031	149	849254	447	150746	45
16	761464	298	911942	149	849522	447	150478	44
17	761642	297	911853	149	849790	446	150210	43
18	761821	297	911763	149	850057	446	149943	42
19	761999	297	911674	149	850325	446	149675	41
20	762177	297	911584	149	850593	446	149407	40
21	9.762356	296	9.911495	149	9.850861	446	10.149139	39
22	762534	296	911405	149	851129	446	148871	38
23	762712	296	911315	150	851396	446	148604	37
24	762889	296	911226	150	851664	446	148336	36
25	763067	296	911136	150	851931	446	148069	35
26	763245	296	911046	150	852199	446	147801	34
27	763422	296	910956	150	852466	446	147534	33
28	763600	295	910866	150	852733	445	147267	32
29	763777	295	910776	150	853001	445	146999	31
30	763954	295	910686	150	853268	445	146732	30
31	9.764131	295	9.910596	150	9.853535	445	10.146465	29
32	764308	295	910506	150	853802	445	146198	28
33	764485	294	910415	150	854069	445	145931	27
34	764662	294	910325	151	854336	445	145664	26
35	764838	294	910235	151	854603	445	145397	25
36	765015	294	910144	151	854870	445	145130	24
37	765191	294	910054	151	855137	445	144863	23
38	765367	294	909963	151	855404	445	144596	22
39	765544	293	909873	151	855671	444	144329	21
40	765720	293	909782	151	855938	444	144062	20
41	9.765896	293	9.909691	151	9.856204	444	10.143796	19
42	766072	293	909601	151	856471	444	143529	18
43	766247	293	909510	151	856737	444	143263	17
44	766423	293	909419	151	857004	444	142996	16
45	766598	292	909328	152	857270	444	142730	15
46	766774	292	909237	152	857537	444	142463	14
47	766949	292	909146	152	857803	444	142197	13
48	767124	292	909055	152	858069	444	141931	12
49	767300	292	908964	152	858336	444	141664	11
50	767475	291	908873	152	858602	443	141398	10
51	9.767649	291	9.908781	152	9.858868	443	10.141132	9
52	767824	291	908690	152	859134	443	140866	8
53	767999	291	908599	152	859400	443	140600	7
54	768173	291	908507	152	859666	443	140334	6
55	768348	290	908416	153	859932	443	140068	5
56	768522	290	908324	153	860198	443	139802	4
57	768697	290	908233	153	860464	443	139536	3
58	768871	290	908141	153	860730	443	139270	2
59	769045	290	908049	153	860995	443	139005	1
60	769219	290	907958	153	861261	443	138739	0
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/
125°								54°

LOGARITHMIC SINES, TANGENTS, ETC.								TABLE II.
36°								143°
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/
0	9.769219	290	9.907958	153	9.861261	443	10.138739	60
1	769393	289	907866	153	861527	443	138473	59
2	769566	289	907774	153	861792	442	138208	58
3	769740	289	907682	153	862058	442	137942	57
4	769913	289	907590	153	862323	442	137677	56
5	770087	289	907498	153	862589	442	137411	55
6	770260	288	907406	153	862854	442	137146	55
7	770433	288	907314	154	863119	442	136881	53
8	770606	288	907222	154	863385	442	136615	52
9	770779	288	907129	154	863650	442	136350	51
10	770952	288	907037	144	863915	442	136085	50
11	9.771125	288	9.906945	154	9.864180	442	10.135820	49
12	771298	287	906852	154	864445	442	135555	48
13	771470	287	906760	154	864710	442	135290	47
14	771643	287	906667	154	864975	441	135025	46
15	771815	287	906575	154	865240	441	134760	45
16	771987	287	906482	154	865505	441	134495	44
17	772159	287	906389	155	865770	441	134230	43
18	772331	286	906296	155	866035	441	133965	42
19	772503	286	906204	155	866300	441	133700	41
20	772675	286	906111	155	866564	441	133436	40
21	9.772847	286	9.906018	155	9.866829	441	10.133171	39
22	773018	286	905925	155	867094	441	132906	38
23	773190	286	905832	155	867358	441	132642	37
24	773361	285	905739	155	867623	441	132377	36
25	773533	285	905645	155	867887	441	132113	35
26	773704	285	905552	155	868152	440	131848	34
27	773875	285	905459	155	868416	440	131584	33
28	774046	285	905366	156	868680	440	131320	32
29	774217	285	905272	156	868945	440	131055	31
30	774388	284	905179	156	869209	440	130791	30
31	9.774558	284	9.905085	156	9.869473	440	10.130527	29
32	774729	284	904992	156	869737	440	130263	28
33	774899	284	904898	156	870001	440	129999	27
34	775070	284	904804	156	870265	440	129735	26
35	775240	284	904711	156	870529	440	129471	25
36	775410	283	904617	156	870793	440	129207	24
37	775580	283	904523	156	871057	440	128943	23
38	775750	283	904429	157	871321	440	128679	22
39	775920	283	904335	157	871585	440	128415	21
40	776090	283	904241	157	871849	439	128151	20
41	9.776259	283	9.904147	157	9.872112	439	10.127888	19
42	776429	282	904053	157	872376	439	127624	18
43	776598	282	903959	157	872640	439	127360	17
44	776768	282	903864	157	872903	439	127097	16
45	776937	282	903770	157	873167	439	126833	15
46	777106	282	903676	157	873430	439	126570	14
47	777275	281	903581	157	873694	439	126306	13
48	777444	281	903487	157	873957	439	126043	12
49	777613	281	903392	158	874220	439	125780	11
50	777781	281	903298	158	874484	439	125516	10
51	9.777950	281	9.903203	158	9.874747	439	10.125253	9
52	778119	281	903108	158	875010	439	124990	8
53	778287	280	903014	158	875273	438	124727	7
54	778455	280	902919	158	875537	438	124463	6
55	778624	280	902824	158	875800	438	124200	5
56	778792	280	902729	158	876063	438	123937	4
57	778960	280	902634	158	876326	438	123674	3
58	779128	280	902539	159	876589	438	123411	2
59	779295	279	902444	159	876852	438	123148	1
60	779463	279	902349	159	877114	438	122886	0
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/
126°								53°

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.

55

37°								142°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	9.779463	279	9.902349	159	9.877114	438	10.122886	60	
1	779631	279	902253	159	877377	438	122623	59	
2	779798	279	902158	159	877640	438	122360	58	
3	779966	279	902063	159	877903	438	122097	57	
4	780133	279	901967	159	878165	438	121835	56	
5	780300	278	901872	159	878428	438	121572	55	
6	780467	278	901776	159	878691	438	121309	54	
7	780634	278	901681	159	878953	437	121047	53	
8	780801	278	901585	159	879216	437	120784	52	
9	780968	278	901490	159	879478	437	120522	51	
10	781134	278	901394	160	879741	437	120259	50	
11	9.781301	277	9.901298	160	9.880003	437	10.119997	49	
12	781468	277	901202	160	880265	437	119735	48	
13	781634	277	901106	160	880528	437	119472	47	
14	781800	277	901010	160	880790	437	119210	46	
15	781966	277	900914	160	881052	437	118948	45	
16	782132	277	900818	160	881314	437	118686	44	
17	782298	276	900722	160	881577	437	118423	43	
18	782464	276	900626	160	881839	437	118161	42	
19	782630	276	900529	160	882101	437	117899	41	
20	782796	276	900433	161	882363	436	117637	40	
21	9.782961	276	9.900337	161	9.882625	436	10.119375	39	
22	783127	276	900240	161	882887	436	117113	38	
23	783292	275	900144	161	883148	436	116852	37	
24	783458	275	900047	161	883410	436	116590	36	
25	783623	275	899951	161	883672	436	116328	35	
26	783788	275	899854	161	883934	436	116066	34	
27	783953	275	899757	161	884196	436	115804	33	
28	784118	275	899660	161	884457	436	115543	32	
29	784282	274	899564	161	884719	436	115281	31	
30	784447	274	899467	162	884980	436	115020	30	
31	9.784612	274	9.899370	162	9.885242	436	10.114758	29	
32	784776	274	899273	162	885504	436	114496	28	
33	784941	274	899176	162	885765	436	114235	27	
34	785105	274	899078	162	886026	436	113974	26	
35	785269	273	898981	162	886288	436	113712	25	
36	785433	273	898884	162	886549	435	113451	24	
37	785597	273	898787	162	886811	435	113189	23	
38	785761	273	898689	162	887072	435	112928	22	
39	785925	273	898592	162	887333	435	112667	21	
40	786089	273	898494	163	887594	435	112406	20	
41	9.786252	272	9.898397	163	9.887855	435	10.112145	19	
42	786416	272	898299	163	888116	435	111884	18	
43	786579	272	898202	163	888378	435	111622	17	
44	786742	272	898104	163	888639	435	111361	16	
45	786906	272	898006	163	888900	435	111100	15	
46	787069	272	897908	163	889161	435	110839	14	
47	787232	271	897810	163	889421	435	110579	13	
48	787395	271	897712	163	889682	435	110318	12	
49	787557	271	897614	163	889943	435	110057	11	
50	787720	271	897516	163	890204	434	109796	10	
51	9.787883	271	9.897418	164	9.890465	434	10.109535	9	
52	788045	271	897320	164	890725	434	109275	8	
53	788208	271	897222	164	890986	434	109014	7	
54	788370	270	897123	164	891247	434	108753	6	
55	788532	270	897025	164	891507	434	108493	5	
56	788694	270	896926	164	891768	434	108232	4	
57	788856	270	896828	164	892028	434	107972	3	
58	789018	270	896729	164	892289	434	107711	2	
59	789180	270	896631	164	892549	434	107451	1	
60	789342	269	896532	164	892810	434	107190	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	

127°

52°

38°										141°									
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/		/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	9.789342	269	9.896532	164	9.892810	434	10.107190	60		0	9.789304	269	896433	165	893070	434	1066930	59	
1	789304	269	896433	165	893070	434	1066669	58		1	789665	269	896335	165	893331	434	1066409	57	
2	789665	269	896335	165	893331	434	1066149	56		2	789827	269	896236	165	893591	434	1066007	55	
3	789827	269	896236	165	893591	434	1065497	54		3	789988	269	896137	165	893851	434	1064907	53	
4	789988	269	896137	165	893851	434	1064097	52		4	790149	269	896038	165	894111	434	1063589	51	
5	790149	269	896038	165	894111	434	1062899	50		5	790310	268	895939	165	894372	434	1061495	50	
6	790310	268	895939	165	894372	434	1060828	48		6	790471	268	895840	165	894632	433	1059889	55	
7	790471	268	895840	165	894632	433	1058289	53		7	790632	268	895741	165	894892	433	1056928	54	
8	790632	268	895741	165	894892	433	1055489	52		8	790793	268	895641	165	895152	433	1053689	51	
9	790793	268	895641	165	895152	433	1051089	50		9	790954	268	895542	165	895412	433	1048489	50	
10	790954	268	895542	165	895412	433	1045889	50		10	9.791115	268	9.895443	166	9.895672	433	10.104328	49	
11	9.791115	268	9.895443	166	9.895672	433	10.104068	48		11	791275	267	895343	166	895932	433	1038087	47	
12	791275	267	895343	166	895932	433	1035487	46		12	791436	267	895244	166	896192	433	1032887	45	
13	791436	267	895244	166	896192	433	1030287	44		13	791596	267	895145	166	896452	433	1030249	43	
14	791596	267	895145	166	896452	433	1027697	42		14	791757	267	895045	166	896712	433	1025097	41	
15	791757	267	895045	166	896712	433	1022497	40		15	791917	267	894945	166	896971	433	1020507	40	
16	791917	267	894945	166	896971	433	1019907	40		16	792077	267	894846	166	897231	433	1017307	39	
17	792077	267	894846	166	897231	433	1014707	38		17	792237	266	894746	166	897491	433	1012117	37	
18	792237	266	894746	166	897491	433	1009517	36		18	792397	266	894646	166	897751	433	1006927	35	
19	792397	266	894646	166	897751	433	1004327	34		19	792557	266	894546	166	898010	433	1001737	33	
20	792557	266	894546	166	898010	433	1000087	32		20	9.792716	266	9.894446	167	9.898270	433	10.099136	32	
21	9.792716	266	9.894446	167	9.898270	433	9996547	31		21	792876	266	894346	167	898530	433	1006927	30	
22	792876	266	894346	167	898530	433	9993957	30		22	793035	266	894246	167	898789	433	1004327	29	
23	793035	266	894246	167	898789	433	9991367	29		23	793195	265	894146	167	899049	432	1001737	28	
24	793195	265	894146	167	899049	432	9988767	28		24	793354	265	894046	167	899308	432	1000697	27	
25	793354	265	894046	167	899308	432	9986177	27		25	793514	265	893946	167	899568	432	9983587	26	
26	793514	265	893946	167	899568	432	9980997	25		26	793673	265	893846	167	899827	432	9978407	25	
27	793673	265	893846	167	899827	432	9975807	24		27	793832	265	893745	167	900087	432	9973217	23	
28	793832	265	893745	167	900087	432	9970627	22		28	793991	265	893645	167	900346	432	9968037	22	
29	793991	265	893645	167	900346	432	9965447	21		29	794150	264	893544	167	900605	432	9960277	20	
30	794150	264	893544	167	900605	432	9962867	19		30	9.794308	264	9.893444	168	9.900864	432	10.099136	20	
31	9.794308	264	9.893444	168	9.900864	432	10.098876	18		31	794467	264	893343	168	901124	432	1006927	19	
32	794467	264	893343	168	901124	432	1006286	17		32	794626	264	893243	168	901383	432	1004327	18	
33	794626	264	893243	168	901383	432	1001736	16		33	794784	264	893142	168	901642	432	1000697	17	
34	794784	264	893142	168	901642	432	9996546	15		34	794942	264	893041	168	901901	432	9993956	16	
35	794942	264	893041	168	901901	432	9991366	14		35	795101	264	892940	168	902160	432	9988766	15	
36	795101	264	892940	168	902160	432	9986176	13		36	795259	263	892839	168	902420	432	9983586	14	
37	795259	263	892839	168	902420	432	9978406	24		37	795417	263	892739	168	902679	432	9975806	23	
38	795417	263	892739	168	902679	432	9973216	22		38	795575	263	892638	168	902938	432	9970626	22	
39	795575	263	892638	168	902938	432	9968036	21		39	795733	263	892536	168	903197	431	9965446	20	
40	795733	263	892536	168	903197	431	9962866	19		40	9.795891	263	9.892435	169	9.903456	431	10.096544	19	
41	9.795891	263	9.892435	169	9.903456	431	10.096286	18		41	796049	263	892334	169	903714	431	1006926	18	
42	796049	263	892334	169	903714	431	1006286	17		42	796206	263	892233	169	903973	431	1004326	17	
43	796206	263	892233	169	903973	431	1001736	16		43	796364	262	892132	169	904232	431	1000696	16	
44	796364	262	892132	169	904232	431	9996546	15		44	796521	262	892030	169	904491	431	9993956	15	
45	796521	262	892030	169	904491	431	9991366	14		45	796679	262	891929	169	904750	431	9988766	15	
46	796679	262	891929	169	904750	431	9986176	13		46	796836	262	891827	169	905008	431	9983586	14	
47	796836	262	891827	169	905008	431	9978406	12		47	796993	262	891726	169	905267	431	9975806	13	
48	796993	262	891726	169	905267	431	9973216	11		48	797150	261	891624	169	905526	431	9970626	12	
49	797150	261	891624	169	905526	431	9968036	10		49	797307	261	891523	170	905785	431	9965446	11	
50	797307	261	891523	170	905785	431	9962866	9		50	9.797464	261	9.891421	170	9.906043	431	10.093957	9	
51	9.797464	261	9.891421	170	9.906043	431	10.093698	8		51	797621	261	891319	170	906302	431	1006926	8	
52	797621	261	891319	170	906302	431	1006286	7		52	797777	261	891217	170	906560	431	1004326	7	
53	797777	261	891217	170	906560	431	1001736	6		53	797934	261	891115	170	906819	431	1000696	6	
54	797934	261	891115	170	906819	431	9996546	5		54	798091	261	891013	170	907077	431	9993956	5	
55	798091	261	891013	170	907077	431	9991366	4		55	798247	261	890911	170	907336	431	9988766	4	
56	798247	261	890911	170	907336	431	9986176	3		56	798403	260	890809	170	907594	431	9983586	3	
57	798403	260	890809	170	907594	431	9978406	2		57	798560	260	890707	170	907853	431	9975806	2	
58	798560	260	890707	170	907853	431	9973216	1		58	798716	260	890605	170	908111	430	9970626	1	
59	798716	260	890605	170	908111	430	9968036	0		59	798872	260	890503	170	908369	430	9965446	0	
60	798872	260	890503	170	908369	430	9962866			60									
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/		/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	
128°										61°									

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.

57

39°								140°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	9.798872	260	9.890503	170	9.908369	430	10.091631	60	
1	799028	260	890400	171	908628	430	091372	59	
2	799184	260	890298	171	908886	430	091114	58	
3	799339	259	890195	171	909144	430	090856	57	
4	799495	259	890093	171	909402	430	090598	56	
5	799651	259	889990	171	909660	430	090340	55	
6	799806	259	889888	171	909918	430	090082	54	
7	799962	259	889785	171	910177	430	089823	53	
8	800117	259	889682	171	910435	430	089565	52	
9	800272	258	889579	171	910693	430	089307	51	
10	800427	258	889477	171	910951	430	089049	50	
11	9.800582	258	9.889374	172	9.911209	430	10.088791	49	
12	800737	258	889271	172	911467	430	088533	48	
13	800892	258	889168	172	911725	430	088275	47	
14	801047	258	889064	172	911982	430	088018	46	
15	801201	258	888961	172	912240	430	087760	45	
16	801356	257	888858	172	912498	430	087502	44	
17	801511	257	888755	172	912756	430	087244	43	
18	801665	257	888651	172	913014	429	086986	42	
19	801819	257	888548	172	913271	429	086729	41	
20	801973	257	888444	173	913529	429	086471	40	
21	9.802128	257	9.888341	173	9.913787	429	10.086213	39	
22	802282	256	888237	173	914044	429	085956	38	
23	802436	256	888134	173	914302	429	085698	37	
24	802589	256	888030	173	914560	429	085440	36	
25	802743	256	887926	173	914817	429	085183	35	
26	802897	256	887822	173	915075	429	084925	34	
27	803050	256	887718	173	915332	429	084668	33	
28	803204	256	887614	173	915590	429	084410	32	
29	803357	255	887510	173	915847	429	084153	31	
30	803511	255	887406	174	916104	429	083896	30	
31	9.803664	255	9.887302	174	9.916362	429	10.083638	29	
32	803817	255	887198	174	916619	429	083381	28	
33	803970	255	887093	174	916877	429	083123	27	
34	804123	255	886989	174	917134	429	082866	26	
35	804276	254	886885	174	917391	429	082609	25	
36	804428	254	886780	174	917648	429	082352	24	
37	804581	254	886676	174	917906	429	082094	23	
38	804734	254	886571	174	918163	428	081837	22	
39	804886	254	886466	174	918420	428	081580	21	
40	805039	254	886362	175	918677	428	081323	20	
41	9.805191	254	9.886257	175	9.918934	428	10.081066	19	
42	805343	253	886152	175	919191	428	080809	18	
43	805495	253	886047	175	919448	428	080552	17	
44	805647	253	885942	175	919705	428	080295	16	
45	805799	253	885837	175	919962	428	080038	15	
46	805951	253	885732	175	920219	428	079781	14	
47	806103	253	885627	175	920476	428	079524	13	
48	806254	253	885522	175	920733	428	079267	12	
49	806406	252	885416	175	920990	428	079010	11	
50	806557	252	885311	176	921247	428	078753	10	
51	9.806709	252	9.885205	176	9.921503	428	10.078497	9	
52	806860	252	885100	176	921760	428	078240	8	
53	807011	252	884994	176	922017	428	077983	7	
54	807163	252	884889	176	922274	428	077726	6	
55	807314	252	884783	176	922530	428	077470	5	
56	807465	251	884677	176	922787	428	077213	4	
57	807615	251	884572	176	923044	428	076956	3	
58	807766	251	884466	176	923300	428	076700	2	
59	807917	251	884360	176	923557	427	076443	1	
60	808067	251	884254	177	923814	427	076186	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	
129°								50°	

40°								139°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	9.808067	251	9.884254	177	9.923814	427	10.076186	60	
1	808218	251	884148	177	924070	427	075930	59	
2	808368	251	884042	177	924327	427	075673	58	
3	808519	250	883936	177	924583	427	075417	57	
4	808669	250	883829	177	924840	427	075160	56	
5	808819	250	883723	177	925096	427	074904	55	
6	808969	250	883617	177	925352	427	074648	54	
7	809119	250	883510	177	925609	427	074391	53	
8	809269	250	883404	177	925865	427	074135	52	
9	809419	249	883297	178	926122	427	073878	51	
10	809569	249	883191	178	926378	427	073622	50	
11	9.809718	249	9.883084	178	9.926634	427	10.073366	49	
12	809868	249	882977	178	926890	427	073110	48	
13	810017	249	882871	178	927147	427	072853	47	
14	810167	249	882764	178	927403	427	072597	46	
15	810316	248	882657	178	927659	427	072341	45	
16	810465	248	882550	178	927915	427	072085	44	
17	810614	248	882443	178	928171	427	071829	43	
18	810763	248	882336	179	928427	427	071573	42	
19	810912	248	882229	179	928684	427	071316	41	
20	811061	248	882121	179	928940	427	071060	40	
21	9.811210	248	9.882014	179	9.929196	427	10.070804	39	
22	811358	247	881907	179	929452	427	070548	38	
23	811507	247	881799	179	929708	427	070292	37	
24	811655	247	881692	179	929964	426	070036	36	
25	811804	247	881584	179	930220	426	069780	35	
26	811952	247	881477	179	930475	426	069525	34	
27	812100	247	881369	179	930731	426	069269	33	
28	812248	247	881261	180	930987	426	069013	32	
29	812396	246	881153	180	931243	426	068757	31	
30	812544	246	881046	180	931499	426	068501	30	
31	9.812692	246	9.880938	180	9.931755	426	10.068245	29	
32	812840	246	880830	180	932010	426	067990	28	
33	812988	246	880722	180	932266	426	067734	27	
34	813135	246	880613	180	932522	426	067478	26	
35	813283	246	880505	180	932778	426	067222	25	
36	813430	245	880397	180	933033	426	066967	24	
37	813578	245	880289	181	933289	426	066711	23	
38	813725	245	880180	181	933545	426	066455	22	
39	813872	245	880072	181	933800	426	066200	21	
40	814019	245	879963	181	934056	426	065944	20	
41	9.814166	245	9.879855	181	9.934311	426	10.065689	19	
42	814313	245	879746	181	934567	426	065433	18	
43	814460	244	879637	181	934822	426	065178	17	
44	814607	244	879529	181	935078	426	064922	16	
45	814753	244	879420	181	935333	426	064667	15	
46	814900	244	879311	181	935589	426	064411	14	
47	815046	244	879202	182	935844	426	064156	13	
48	815193	244	879093	182	936100	426	063900	12	
49	815339	244	878984	182	936355	426	063645	11	
50	815485	243	878875	182	936611	426	063389	10	
51	9.815631	243	9.878766	182	9.936866	425	10.063134	9	
52	815778	243	878656	182	937121	425	062879	8	
53	815924	243	878547	182	937377	425	062623	7	
54	816069	243	878438	182	937632	425	062368	6	
55	816215	243	878328	182	937887	425	062113	5	
56	816361	243	878219	183	938142	425	061858	4	
57	816507	242	878109	183	938398	425	061602	3	
58	816652	242	877999	183	938653	425	061347	2	
59	816798	242	877890	183	938908	425	061092	1	
60	816943	242	877780	183	939163	425	060837	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	

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TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.

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41°								138°							
'	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	'							
0	9.816943	242	9.877780	183	9.939163	425	10.060837	60							
1	817088	242	877670	183	939418	425	060582	59							
2	817233	242	877560	183	939673	425	060327	58							
3	817379	242	877450	183	939928	425	060072	57							
4	817524	241	877340	183	940183	425	059817	56							
5	817668	241	877230	184	940439	425	059561	55							
6	817813	241	877120	184	940694	425	059306	54							
7	817958	241	877010	184	940949	425	059051	53							
8	818103	241	876899	184	941204	425	058796	52							
9	818247	241	876789	184	941459	425	058541	51							
10	818392	241	876678	184	941713	425	058287	50							
11	9.818536	240	9.876568	184	9.941968	425	10.058632	49							
12	818681	240	876457	184	942223	425	057777	48							
13	818825	240	876347	184	942478	425	057522	47							
14	818969	240	876236	185	942733	425	057267	46							
15	819113	240	876125	185	942988	425	057012	45							
16	819257	240	876014	185	943243	425	056757	44							
17	819401	240	875904	185	943498	425	056502	43							
18	819545	239	875793	185	943752	425	056248	42							
19	819689	239	875682	185	944007	425	055993	41							
20	819832	239	875571	185	944262	425	055738	40							
21	9.819976	239	9.875459	185	9.944517	425	10.055483	39							
22	820120	239	875348	185	944771	424	055229	38							
23	820263	239	875237	185	945026	424	054974	37							
24	820406	239	875126	186	945281	424	054719	36							
25	820550	238	875014	186	945535	424	054465	35							
26	820693	238	874903	186	945790	424	054210	34							
27	820836	238	874791	186	946045	424	053955	33							
28	820979	238	874680	186	946299	424	053701	32							
29	821122	238	874568	186	946554	424	053446	31							
30	821265	238	874456	186	946808	424	053192	30							
31	9.821407	238	9.874344	186	9.947063	424	10.052937	29							
32	821550	238	874232	187	947318	424	052682	28							
33	821693	237	874121	187	947572	424	052428	27							
34	821835	237	874009	187	947827	424	052173	26							
35	821977	237	873896	187	948081	424	051919	25							
36	822120	237	873784	187	948335	424	051665	24							
37	822262	237	873672	187	948590	424	051410	23							
38	822404	237	873560	187	948844	424	051156	22							
39	822546	237	873448	187	949099	424	050901	21							
40	822688	236	873335	187	949353	424	050647	20							
41	9.822830	236	9.873223	187	9.949608	424	10.050392	19							
42	822972	236	873110	188	949862	424	050138	18							
43	823114	236	872998	188	950116	424	049884	17							
44	823255	236	872885	188	950371	424	049629	16							
45	823397	236	872772	188	950625	424	049375	15							
46	823539	236	872659	188	950879	424	049121	14							
47	823680	235	872547	188	951133	424	048867	13							
48	823821	235	872434	188	951388	424	048612	12							
49	823963	235	872321	188	951642	424	048358	11							
50	824104	235	872208	188	951896	424	048104	10							
51	9.824245	235	9.872095	189	9.952150	424	10.047850	9							
52	824386	235	871981	189	952405	424	047595	8							
53	824527	235	871868	189	952659	424	047341	7							
54	824668	234	871755	189	952913	424	047087	6							
55	824808	234	871641	189	953167	423	046833	5							
56	824949	234	871528	189	953421	423	046579	4							
57	825090	234	871414	189	953675	423	046325	3							
58	825230	234	871301	189	953929	423	046071	2							
59	825371	234	871187	189	954183	423	045817	1							
60	825511	234	871073	190	954437	423	045563	0							
'	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	'							

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60		LOGARITHMIC SINES, TANGENTS, ETC.						TABLE II.	
42°								137°	
'	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	'	
0	9.825511	234	9.871073	190	9.954437	423	10.045563	60	
1	825651	233	870960	190	954691	423	045309	59	
2	825791	233	870846	190	954946	423	045054	58	
3	825931	233	870732	190	955200	423	044800	57	
4	826071	233	870618	190	955454	423	044546	56	
5	826211	233	870504	190	955708	423	044292	55	
6	826351	233	870390	190	955961	423	044039	54	
7	826491	233	870276	190	956215	423	043785	53	
8	826631	233	870161	190	956469	423	043531	52	
9	826770	232	870047	191	956723	423	043277	51	
10	826910	232	869933	191	956977	423	043023	50	
11	9.827049	232	9.869818	191	9.957231	423	10.042769	49	
12	827189	232	869704	191	957485	423	042515	48	
13	827328	232	869589	191	957739	423	042261	47	
14	827467	232	869474	191	957993	423	042007	46	
15	827606	232	869360	191	958247	423	041753	45	
16	827745	232	869245	191	958500	423	041500	44	
17	827884	231	869130	191	958754	423	041246	43	
18	828023	231	869015	192	959008	423	040992	42	
19	828162	231	868900	192	959262	423	040738	41	
20	828301	231	868785	192	959516	423	040484	40	
21	9.828439	231	9.868670	192	9.959769	423	10.040231	39	
22	828578	231	868555	192	960023	423	039977	38	
23	828716	231	868440	192	960277	423	039723	37	
24	828855	230	868324	192	960530	423	039470	36	
25	828993	230	868209	192	960784	423	039216	35	
26	829131	230	868093	192	961038	423	038962	34	
27	829269	230	867978	193	961292	423	038708	33	
28	829407	230	867862	193	961545	423	038455	32	
29	829545	230	867747	193	961799	423	038201	31	
30	829683	230	867631	193	962052	423	037948	30	
31	9.829821	229	9.867515	193	9.962306	423	10.037694	29	
32	829959	229	867399	193	962560	423	037440	28	
33	830097	229	867283	193	962813	423	037187	27	
34	830234	229	867167	193	963067	423	036933	26	
35	830372	229	867051	193	963320	423	036680	25	
36	830509	229	866935	194	963574	423	036426	24	
37	830646	229	866819	194	963828	423	036172	23	
38	830784	229	866703	194	964081	423	035919	22	
39	830921	228	866586	194	964335	423	035665	21	
40	831058	228	866470	194	964588	422	035412	20	
41	9.831195	228	9.866353	194	9.964842	422	10.035158	19	
42	831332	228	866237	194	965095	422	034905	18	
43	831469	228	866120	194	965349	422	034651	17	
44	831606	228	866004	195	965602	422	034398	16	
45	831742	228	865887	195	965855	422	034145	15	
46	831879	228	865770	195	966109	422	033891	14	
47	832015	227	865653	195	966362	422	033638	13	
48	832152	227	865536	195	966616	422	033384	12	
49	832288	227	865419	195	966869	422	033131	11	
50	832425	227	865302	195	967123	422	032877	10	
51	9.832561	227	9.865185	195	9.967376	422	10.032624	9	
52	832697	227	865068	195	967629	422	032371	8	
53	832833	227	864950	195	967883	422	032117	7	
54	832969	226	864833	196	968136	422	031864	6	
55	833105	226	864716	196	968389	422	031611	5	
56	833241	226	864598	196	968643	422	031357	4	
57	833377	226	864481	196	968896	422	031104	3	
58	833512	226	864363	196	969149	422	030851	2	
59	833648	226	864245	196	969403	422	030597	1	
60	833783	226	864127	196	969656	422	030344	0	
'	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	'	
132°								47°	

TABLE II. LOGARITHMIC SINES, TANGENTS, ETC.

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43°								136°	
/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/	
0	9.833783	226	9.864127	196	9.966656	422	10.030344	60	
1	833919	225	864010	196	969909	422	030091	59	
2	834054	225	863892	197	970162	422	029838	58	
3	834189	225	863774	197	970416	422	029584	57	
4	834325	225	863656	197	970669	422	029331	56	
5	834460	225	863538	197	970922	422	029078	55	
6	834595	225	863419	197	971175	422	028825	54	
7	834730	225	863301	197	971429	422	028571	53	
8	834865	225	863183	197	971682	422	028318	52	
9	834999	224	863064	197	971935	422	028065	51	
10	835134	224	862946	198	972188	422	027812	50	
11	9.835269	224	9.862827	198	9.972441	422	10.027559	49	
12	835403	224	862709	198	972695	422	027305	48	
13	835538	224	862590	198	972948	422	027052	47	
14	835672	224	862471	198	973201	422	026799	46	
15	835807	224	862353	198	973454	422	026546	45	
16	835941	224	862234	198	973707	422	026293	44	
17	836075	223	862115	198	973960	422	026040	43	
18	836209	223	861996	198	974213	422	025787	42	
19	836343	223	861877	198	974466	422	025534	41	
20	836477	223	861758	199	974720	422	025280	40	
21	9.836611	223	9.861638	199	9.974973	422	10.025027	39	
22	836745	223	861519	199	975226	422	024774	38	
23	836878	223	861400	199	975479	422	024521	37	
24	837012	222	861280	199	975732	422	024268	36	
25	837146	222	861161	199	975985	422	024015	35	
26	837279	222	861041	199	976238	422	023762	34	
27	837412	222	860922	199	976491	422	023509	33	
28	837546	222	860802	199	976744	422	023256	32	
29	837679	222	860682	200	976997	422	023003	31	
30	837812	222	860562	200	977250	422	022750	30	
31	9.837945	222	9.860442	200	9.977503	422	10.022497	29	
32	838078	221	860322	200	977756	422	022244	28	
33	838211	221	860202	200	978009	422	021991	27	
34	838344	221	860082	200	978262	422	021738	26	
35	838477	221	859962	200	978515	422	021485	25	
36	838610	221	859842	200	978768	422	021232	24	
37	838742	221	859721	201	979021	422	020979	23	
38	838875	221	859601	201	979274	422	020726	22	
39	839007	221	859480	201	979527	422	020473	21	
40	839140	220	859360	201	979780	422	020220	20	
41	9.839272	220	9.859239	201	9.980033	422	10.019967	19	
42	839404	220	859119	201	980286	422	019714	18	
43	839536	220	858998	201	980538	422	019462	17	
44	839668	220	858877	201	980791	421	019209	16	
45	839800	220	858756	202	981044	421	018956	15	
46	839932	220	858635	202	981297	421	018703	14	
47	840064	219	858514	202	981550	421	018450	13	
48	840196	219	858393	202	981803	421	018197	12	
49	840328	219	858272	202	982056	421	017944	11	
50	840459	219	858151	202	982309	421	017691	10	
51	9.840591	219	9.858029	202	9.982562	421	10.017438	9	
52	840722	219	857908	202	982814	421	017186	8	
53	840854	219	857786	202	983067	421	016933	7	
54	840985	219	857665	203	983320	421	016680	6	
55	841116	218	857543	203	983573	421	016427	5	
56	841247	218	857422	203	983826	421	016174	4	
57	841378	218	857300	203	984079	421	015921	3	
58	841509	218	857178	203	984332	421	015668	2	
59	841640	218	857056	203	984584	421	015416	1	
60	841771	218	856934	203	984837	421	015163	0	
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/	

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135°

/	Sine.	D.	Cosine.	D.	Tang.	D.	Cotang.	/
0	9.841771	218	9.856934	203	9.984837	421	10.015163	60
1	841902	218	856812	203	985090	421	014910	59
2	842033	218	856690	204	985343	421	014657	58
3	842163	217	856568	204	985596	421	014404	57
4	842294	217	856446	204	985848	421	014152	56
5	842424	217	856323	204	986101	421	013899	55
6	842555	217	856201	204	986354	421	013646	54
7	842685	217	856078	204	986607	421	013393	53
8	842815	217	855956	204	986860	421	013140	52
9	842946	217	855833	204	987112	421	012888	51
10	843076	217	855711	205	987365	421	012635	50
11	9.843206	216	9.855588	205	9.987618	421	10.012382	49
12	843336	216	855465	205	987871	421	012129	48
13	843466	216	855342	205	988123	421	011877	47
14	843595	216	855219	205	988376	421	011624	46
15	843725	216	855096	205	988629	421	011371	45
16	843855	216	854973	205	988882	421	011118	44
17	843984	216	854850	205	989134	421	010866	43
18	844114	215	854727	206	989387	421	010613	42
19	844243	215	854603	206	989640	421	010360	41
20	844372	215	854480	206	989893	421	010107	40
21	9.844502	215	9.854356	206	9.990145	421	10.009855	39
22	844631	215	854233	206	990398	421	009602	38
23	844760	215	854109	206	990651	421	009349	37
24	844889	215	853986	206	990903	421	009097	36
25	845018	215	853862	206	991156	421	008844	35
26	845147	215	853738	206	991409	421	008591	34
27	845276	214	853614	207	991662	421	008338	33
28	845405	214	853490	207	991914	421	008086	32
29	845533	214	853366	207	992167	421	007833	31
30	845662	214	853242	207	992420	421	007580	30
31	9.845790	214	9.853118	207	9.992672	421	10.007328	29
32	845919	214	852994	207	992925	421	007075	28
33	846047	214	852869	207	993178	421	006822	27
34	846175	214	852745	207	993431	421	006569	26
35	846304	214	852620	207	993683	421	006317	25
36	846432	213	852496	208	993936	421	006064	24
37	846560	213	852371	208	994189	421	005811	23
38	846688	213	852247	208	994441	421	005559	22
39	846816	213	852122	208	994694	421	005306	21
40	846944	213	851997	208	994947	421	005053	20
41	9.847071	213	9.851872	208	9.995199	421	10.004801	19
42	847199	213	851747	208	995452	421	004548	18
43	847327	213	851622	208	995705	421	004295	17
44	847454	212	851497	209	995957	421	004043	16
45	847582	212	851372	209	996210	421	003790	15
46	847709	212	851246	209	996463	421	003537	14
47	847836	212	851121	209	996715	421	003285	13
48	847964	212	850996	209	996968	421	003032	12
49	848091	212	850870	209	997221	421	002779	11
50	848218	212	850745	209	997473	421	002527	10
51	9.848345	212	9.850619	209	9.997726	421	10.002274	9
52	848472	211	850493	210	997979	421	002021	8
53	848599	211	850368	210	998231	421	001769	7
54	848726	211	850242	210	998484	421	001516	6
55	848852	211	850116	210	998737	421	001263	5
56	848979	211	849990	210	998989	421	001011	4
57	849106	211	849864	210	999242	421	000758	3
58	849232	211	849738	210	999495	421	000505	2
59	849359	211	849611	210	999747	421	000253	1
60	849485	211	849485	210	10.000000	421	10.000000	0
/	Cosine.	D.	Sine.	D.	Cotang.	D.	Tang.	/

134°

45°

T A B L E III.,

OF

N A T U R A L S I N E S A N D T A N G E N T S ;

TO

E V E R Y D E G R E E A N D M I N U T E O F T H E Q U A D R A N T .



If the given angle is less than 45° , look for the degrees and the title of the column, at the *top* of the page ; and for the minutes on the *left*. But if the angle is between 45° and 90° , look for the degrees and the title of the column, at the *bottom* ; and for the minutes on the *right*.

The *Secants and Cosecants*, which are not inserted in this table, may be easily supplied. If 1 be divided by the cosine of an arc, the quotient will be the secant of that arc. And if 1 be divided by the sine, the quotient will be the cosecant.

The values of the Sines and Cosines are less than a unit, and are given in decimals, although the decimal point is not printed. So also, the tangents of arcs less than 45° , and cotangents of arcs greater than 45° , are less than a unit and are expressed in decimals with the decimal point omitted.

64		NATURAL SINES AND COSINES.										TABLE III.	
/	0°		1°		2°		3°		4°		/		
	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.			
0	00000	Unit.	01745	99985	03490	99939	05234	99863	06976	99756	60		
1	00029	Unit.	01774	99984	03519	99938	05263	99861	07005	99754	59		
2	00058	Unit.	01803	99984	03548	99937	05292	99860	07034	99752	58		
3	00087	Unit.	01832	99983	03577	99936	05321	99858	07063	99750	57		
4	00116	Unit.	01862	99983	03606	99935	05350	99857	07092	99748	56		
5	00145	Unit.	01891	99982	03635	99934	05379	99855	07121	99746	55		
6	00175	Unit.	01920	99982	03664	99933	05408	99854	07150	99744	54		
7	00204	Unit.	01949	99981	03693	99932	05437	99852	07179	99742	53		
8	00233	Unit.	01978	99980	03723	99931	05466	99851	07208	99740	52		
9	00262	Unit.	02007	99980	03752	99930	05495	99849	07237	99738	51		
10	00291	Unit.	02036	99979	03781	99929	05524	99847	07266	99736	50		
11	00320	99999	02065	99979	03810	99927	05553	99846	07295	99734	49		
12	00349	99999	02094	99978	03839	99926	05582	99844	07324	99731	48		
13	00378	99999	02123	99977	03868	99925	05611	99842	07353	99729	47		
14	00407	99999	02152	99977	03897	99924	05640	99841	07382	99727	46		
15	00436	99999	02181	99976	03926	99923	05669	99839	07411	99725	45		
16	00465	99999	02211	99976	03955	99922	05698	99838	07440	99723	44		
17	00495	99999	02240	99975	03984	99921	05727	99836	07469	99721	43		
18	00524	99999	02269	99974	04013	99919	05756	99834	07498	99719	42		
19	00553	99998	02298	99974	04042	99918	05785	99833	07527	99716	41		
20	00582	99998	02327	99973	04071	99917	05814	99831	07556	99714	40		
21	00611	99998	02356	99972	04100	99916	05844	99829	07585	99712	39		
22	00640	99998	02385	99972	04129	99915	05873	99827	07614	99710	38		
23	00669	99998	02414	99971	04159	99913	05902	99826	07643	99708	37		
24	00698	99998	02443	99970	04188	99912	05931	99824	07672	99705	36		
25	00727	99997	02472	99969	04217	99911	05960	99822	07701	99703	35		
26	00756	99997	02501	99969	04246	99910	05989	99821	07730	99701	34		
27	00785	99997	02530	99968	04275	99909	06018	99819	07759	99699	33		
28	00814	99997	02560	99967	04304	99907	06047	99817	07788	99696	32		
29	00844	99996	02589	99966	04333	99906	06076	99815	07817	99694	31		
30	00873	99996	02618	99966	04362	99905	06105	99813	07846	99692	30		
31	00902	99996	02647	99965	04391	99904	06134	99812	07875	99689	29		
32	00931	99996	02676	99964	04420	99902	06163	99810	07904	99687	28		
33	00960	99995	02705	99963	04449	99901	06192	99808	07933	99685	27		
34	00989	99995	02734	99963	04478	99900	06221	99806	07962	99683	26		
35	01018	99995	02763	99962	04507	99898	06250	99804	07991	99680	25		
36	01047	99995	02792	99961	04536	99897	06279	99803	08020	99678	24		
37	01076	99994	02821	99960	04565	99896	06308	99801	08049	99676	23		
38	01105	99994	02850	99959	04594	99894	06337	99799	08078	99673	22		
39	01134	99994	02879	99959	04623	99893	06366	99797	08107	99671	21		
40	01164	99993	02908	99958	04653	99892	06395	99795	08136	99668	20		
41	01193	99993	02938	99957	04682	99890	06424	99793	08165	99666	19		
42	01222	99993	02967	99956	04711	99889	06453	99792	08194	99664	18		
43	01251	99992	02996	99955	04740	99888	06482	99790	08223	99661	17		
44	01280	99992	03025	99954	04769	99886	06511	99788	08252	99659	16		
45	01309	99991	03054	99953	04798	99885	06540	99786	08281	99657	15		
46	01338	99991	03083	99952	04827	99883	06569	99784	08310	99654	14		
47	01367	99991	03112	99952	04856	99882	06598	99782	08339	99652	13		
48	01396	99990	03141	99951	04885	99881	06627	99780	08368	99649	12		
49	01425	99990	03170	99950	04914	99879	06656	99778	08397	99647	11		
50	01454	99989	03199	99949	04943	99878	06685	99776	08426	99644	10		
51	01483	99989	03228	99948	04972	99876	06714	99774	08455	99642	9		
52	01513	99989	03257	99947	05001	99875	06743	99772	08484	99639	8		
53	01542	99988	03286	99946	05030	99873	06773	99770	08513	99637	7		
54	01571	99988	03316	99945	05059	99872	06802	99768	08542	99635	6		
55	01600	99987	03345	99944	05088	99870	06831	99766	08571	99632	5		
56	01629	99987	03374	99943	05117	99869	06860	99764	08600	99630	4		
57	01658	99986	03403	99942	05146	99867	06889	99762	08629	99627	3		
58	01687	99986	03432	99941	05175	99866	06918	99760	08658	99625	2		
59	01716	99985	03461	99940	05205	99864	06947	99758	08687	99622	1		
60	01745	99985	03490	99939	05234	99863	06976	99756	08716	99619	0		
	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.			
	89°		88°		87°		86°		85°				

TABLE III.

NATURAL SINES AND COSINES.

65

°	5°		6°		7°		8°		9°		°
	Sine	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	
0	08716	99619	10453	99452	12187	99255	13917	99027	15643	98769	60
1	08745	99617	10482	99449	12216	99251	13946	99023	15672	98764	59
2	08774	99614	10511	99446	12245	99248	13975	99019	15701	98760	58
3	08803	99612	10540	99443	12274	99244	14004	99015	15730	98755	57
4	08831	99609	10569	99440	12302	99240	14033	99011	15758	98751	56
5	08860	99607	10597	99437	12331	99237	14061	99006	15787	98746	55
6	08889	99604	10626	99434	12360	99233	14090	99002	15816	98741	54
7	08918	99602	10655	99431	12389	99230	14119	98998	15845	98737	53
8	08947	99599	10684	99428	12418	99226	14148	98994	15873	98732	52
9	08976	99596	10713	99424	12447	99222	14177	98990	15902	98728	51
10	09005	99594	10742	99421	12476	99219	14205	98986	15931	98723	50
11	09034	99591	10771	99418	12504	99215	14234	98982	15959	98718	49
12	09063	99588	10800	99415	12533	99211	14263	98978	15988	98714	48
13	09092	99586	10829	99412	12562	99208	14292	98973	16017	98709	47
14	09121	99583	10858	99409	12591	99204	14320	98969	16046	98704	46
15	09150	99580	10887	99406	12620	99200	14349	98965	16074	98700	45
16	09179	99578	10916	99402	12649	99197	14378	98961	16103	98695	44
17	09208	99575	10945	99399	12678	99193	14407	98957	16132	98690	43
18	09237	99572	10973	99396	12706	99189	14436	98953	16160	98686	42
19	09266	99570	11002	99393	12735	99186	14464	98948	16189	98681	41
20	09295	99567	11031	99390	12764	99182	14493	98944	16218	98676	40
21	09324	99564	11060	99386	12793	99178	14522	98940	16246	98671	39
22	09353	99562	11089	99383	12822	99175	14551	98936	16275	98667	38
23	09382	99559	11118	99380	12851	99171	14580	98931	16304	98662	37
24	09411	99556	11147	99377	12880	99167	14608	98927	16333	98657	36
25	09440	99553	11176	99374	12908	99163	14637	98923	16361	98652	35
26	09469	99551	11205	99370	12937	99160	14666	98919	16390	98648	34
27	09498	99548	11234	99367	12966	99156	14695	98914	16419	98643	33
28	09527	99545	11263	99364	12995	99152	14723	98910	16447	98638	32
29	09556	99542	11291	99360	13024	99148	14752	98906	16476	98633	31
30	09585	99540	11320	99357	13053	99144	14781	98902	16505	98629	30
31	09614	99537	11349	99354	13081	99141	14810	98897	16533	98624	29
32	09642	99534	11378	99351	13110	99137	14838	98893	16562	98619	28
33	09671	99531	11407	99347	13139	99133	14867	98889	16591	98614	27
34	09700	99528	11436	99344	13168	99129	14896	98884	16620	98609	26
35	09729	99526	11465	99341	13197	99125	14925	98880	16648	98604	25
36	09758	99523	11494	99337	13226	99122	14954	98876	16677	98600	24
37	09787	99520	11523	99334	13254	99118	14982	98871	16706	98595	23
38	09816	99517	11552	99331	13283	99114	15011	98867	16734	98590	22
39	09845	99514	11580	99327	13312	99110	15040	98863	16763	98585	21
40	09874	99511	11609	99324	13341	99106	15069	98858	16792	98580	20
41	09903	99508	11638	99320	13370	99102	15097	98854	16820	98575	19
42	09932	99506	11667	99317	13399	99098	15126	98849	16849	98570	18
43	09961	99503	11696	99314	13427	99094	15155	98845	16878	98565	17
44	09990	99500	11725	99310	13456	99091	15184	98841	16906	98561	16
45	10019	99497	11754	99307	13485	99087	15212	98836	16935	98556	15
46	10048	99494	11783	99303	13514	99083	15241	98832	16964	98551	14
47	10077	99491	11812	99300	13543	99079	15270	98827	16992	98546	13
48	10106	99488	11840	99297	13572	99075	15299	98823	17021	98541	12
49	10135	99485	11869	99293	13600	99071	15327	98818	17050	98536	11
50	10164	99482	11898	99290	13629	99067	15356	98814	17078	98531	10
51	10192	99479	11927	99286	13658	99063	15385	98809	17107	98526	9
52	10221	99476	11956	99283	13687	99059	15414	98805	17136	98521	8
53	10250	99473	11985	99279	13716	99055	15442	98800	17164	98516	7
54	10279	99470	12014	99276	13744	99051	15471	98796	17193	98511	6
55	10308	99467	12043	99272	13773	99047	15500	98791	17222	98506	5
56	10337	99464	12071	99269	13802	99043	15529	98787	17250	98501	4
57	10366	99461	12100	99265	13831	99039	15557	98782	17279	98496	3
58	10395	99458	12129	99262	13860	99035	15586	98778	17308	98491	2
59	10424	99455	12158	99258	13889	99031	15615	98773	17336	98486	1
60	10453	99452	12187	99255	13917	99027	15643	98769	17365	98481	0
°	84°		83°		82°		81°		80°		°
	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	

°	10°		11°		12°		13°		14°		°
	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	
0	17365	98481	19081	98163	20791	97815	22495	97437	24192	97030	60
1	17393	98476	19100	98157	20820	97809	22523	97430	24220	97023	59
2	17422	98471	19138	98152	20848	97803	22552	97424	24249	97015	58
3	17451	98466	19167	98146	20877	97797	22580	97417	24277	97008	57
4	17479	98461	19195	98140	20905	97791	22608	97411	24305	97001	56
5	17508	98455	19224	98135	20933	97784	22637	97404	24333	96994	55
6	17537	98450	19252	98129	20962	97778	22665	97398	24362	96987	54
7	17565	98445	19281	98124	20990	97772	22693	97391	24390	96980	53
8	17594	98440	19309	98118	21019	97766	22722	97384	24418	96973	52
9	17623	98435	19338	98112	21047	97760	22750	97378	24446	96966	51
10	17651	98430	19366	98107	21076	97754	22778	97371	24474	96959	50
11	17680	98425	19395	98101	21104	97748	22807	97365	24503	96952	49
12	17708	98420	19423	98096	21132	97742	22835	97358	24531	96945	48
13	17737	98414	19452	98090	21161	97735	22863	97351	24559	96937	47
14	17766	98409	19481	98084	21189	97729	22892	97345	24587	96930	46
15	17794	98404	19509	98079	21218	97723	22920	97338	24615	96923	45
16	17823	98399	19538	98073	21246	97717	22948	97331	24644	96916	44
17	17852	98394	19566	98067	21275	97711	22977	97325	24672	96909	43
18	17880	98389	19595	98061	21303	97705	23005	97318	24700	96902	42
19	17909	98383	19623	98056	21331	97698	23033	97311	24728	96894	41
20	17937	98378	19652	98050	21360	97692	23062	97304	24756	96887	40
21	17966	98373	19680	98044	21388	97686	23090	97298	24784	96880	39
22	17995	98368	19709	98039	21417	97680	23118	97291	24813	96873	38
23	18023	98362	19737	98033	21445	97673	23146	97284	24841	96866	37
24	18052	98357	19766	98027	21474	97667	23175	97278	24869	96858	36
25	18081	98352	19794	98021	21502	97661	23203	97271	24897	96851	35
26	18109	98347	19823	98016	21530	97655	23231	97264	24925	96844	34
27	18138	98341	19851	98010	21559	97648	23260	97257	24954	96837	33
28	18166	98336	19880	98004	21587	97642	23288	97251	24982	96829	32
29	18195	98331	19908	97998	21616	97636	23316	97244	25010	96822	31
30	18224	98325	19937	97992	21644	97630	23345	97237	25038	96815	30
31	18252	98320	19965	97987	21672	97623	23373	97230	25066	96807	29
32	18281	98315	19994	97981	21701	97617	23401	97223	25094	96800	28
33	18309	98310	20022	97975	21729	97611	23429	97217	25122	96793	27
34	18338	98304	20051	97969	21758	97604	23458	97210	25151	96786	26
35	18367	98299	20079	97963	21786	97598	23486	97203	25179	96778	25
36	18395	98294	20108	97958	21814	97592	23514	97196	25207	96771	24
37	18424	98288	20136	97952	21843	97585	23542	97189	25235	96764	23
38	18452	98283	20165	97946	21871	97579	23571	97182	25263	96756	22
39	18481	98277	20193	97940	21899	97573	23599	97176	25291	96749	21
40	18509	98272	20222	97934	21928	97566	23627	97169	25320	96742	20
41	18538	98267	20250	97928	21956	97560	23656	97162	25348	96734	19
42	18567	98261	20279	97922	21985	97553	23684	97155	25376	96727	18
43	18595	98256	20307	97916	22013	97547	23712	97148	25404	96719	17
44	18624	98250	20336	97910	22041	97541	23740	97141	25432	96712	16
45	18652	98245	20364	97905	22070	97534	23769	97134	25460	96705	15
46	18681	98240	20393	97899	22098	97528	23797	97127	25488	96697	14
47	18710	98234	20421	97893	22126	97521	23825	97120	25516	96690	13
48	18738	98229	20450	97887	22155	97515	23853	97113	25545	96682	12
49	18767	98223	20478	97881	22183	97508	23882	97106	25573	96675	11
50	18795	98218	20507	97875	22212	97502	23910	97100	25601	96667	10
51	18824	98212	20535	97869	22240	97496	23938	97093	25629	96660	9
52	18852	98207	20563	97863	22268	97489	23966	97086	25657	96653	8
53	18881	98201	20592	97857	22297	97483	23995	97079	25685	96645	7
54	18910	98196	20620	97851	22325	97476	24023	97072	25713	96638	6
55	18938	98190	20649	97845	22353	97470	24051	97065	25741	96630	5
56	18967	98185	20677	97839	22382	97463	24079	97058	25769	96623	4
57	18995	98179	20706	97833	22410	97457	24108	97051	25798	96615	3
58	19024	98174	20734	97827	22438	97450	24136	97044	25826	96608	2
59	19052	98168	20763	97821	22467	97444	24164	97037	25854	96600	1
60	19081	98163	20791	97815	22495	97437	24192	97030	25882	96593	0
°	79°		78°		77°		76°		75°		°
	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	

TABLE III.

NATURAL SINES AND COSINES.

67

'	15°		16°		17°		18°		19°		'
	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	
0	25882	96593	27564	96126	29237	95630	30902	95106	32557	94552	60
1	25910	96585	27592	96118	29265	95622	30929	95097	32584	94542	59
2	25938	96578	27620	96110	29293	95613	30957	95088	32612	94533	58
3	25966	96570	27648	96102	29321	95605	30985	95079	32639	94523	57
4	25994	96562	27676	96094	29348	95596	31012	95070	32667	94514	56
5	26022	96555	27704	96086	29376	95588	31040	95061	32694	94504	55
6	26050	96547	27731	96078	29404	95579	31068	95052	32722	94495	54
7	26079	96540	27759	96070	29432	95571	31095	95043	32749	94485	53
8	26107	96532	27787	96062	29460	95562	31123	95033	32777	94476	52
9	26135	96524	27815	96054	29487	95554	31151	95024	32804	94466	51
10	26163	96517	27843	96046	29515	95545	31178	95015	32832	94457	50
11	26191	96509	27871	96037	29543	95536	31206	95006	32859	94447	49
12	26219	96502	27899	96029	29571	95528	31233	94997	32887	94438	48
13	26247	96494	27927	96021	29599	95519	31261	94988	32914	94428	47
14	26275	96486	27955	96013	29626	95511	31289	94979	32942	94418	46
15	26303	96479	27983	96005	29654	95502	31316	94970	32969	94409	45
16	26331	96471	28011	95997	29682	95493	31344	94961	32997	94399	44
17	26359	96463	28039	95989	29710	95485	31372	94952	33024	94390	43
18	26387	96456	28067	95981	29737	95476	31399	94943	33051	94380	42
19	26415	96448	28095	95972	29765	95467	31427	94933	33079	94370	41
20	26443	96440	28123	95964	29793	95459	31454	94924	33106	94361	40
21	26471	96433	28150	95956	29821	95450	31482	94915	33134	94351	39
22	26500	96425	28178	95948	29849	95441	31510	94906	33161	94342	38
23	26528	96417	28206	95940	29876	95433	31537	94897	33189	94332	37
24	26556	96410	28234	95931	29904	95424	31565	94888	33216	94322	36
25	26584	96402	28262	95923	29932	95415	31593	94879	33244	94313	35
26	26612	96394	28290	95915	29960	95407	31620	94869	33271	94303	34
27	26640	96386	28318	95907	29987	95398	31648	94860	33298	94293	33
28	26668	96379	28346	95898	30015	95389	31675	94851	33326	94284	32
29	26696	96371	28374	95890	30043	95380	31703	94842	33353	94274	31
30	26724	96363	28402	95882	30071	95372	31730	94832	33381	94264	30
31	26752	96355	28429	95874	30098	95363	31758	94823	33408	94254	29
32	26780	96347	28457	95865	30126	95354	31786	94814	33436	94245	28
33	26808	96340	28485	95857	30154	95345	31813	94805	33463	94235	27
34	26836	96332	28513	95849	30182	95337	31841	94795	33490	94225	26
35	26864	96324	28541	95841	30209	95328	31868	94786	33518	94215	25
36	26892	96316	28569	95832	30237	95319	31896	94777	33545	94206	24
37	26920	96308	28597	95824	30265	95310	31923	94768	33573	94196	23
38	26948	96301	28625	95816	30292	95301	31951	94758	33600	94186	22
39	26976	96293	28652	95807	30320	95293	31979	94749	33627	94176	21
40	27004	96285	28680	95799	30348	95284	32006	94740	33655	94167	20
41	27032	96277	28708	95791	30376	95275	32034	94730	33682	94157	19
42	27060	96269	28736	95782	30403	95266	32061	94721	33710	94147	18
43	27088	96261	28764	95774	30431	95257	32089	94712	33737	94137	17
44	27116	96253	28792	95766	30459	95248	32116	94702	33764	94127	16
45	27144	96246	28820	95757	30486	95240	32144	94693	33792	94118	15
46	27172	96238	28847	95749	30514	95231	32171	94684	33819	94108	14
47	27200	96230	28875	95740	30542	95222	32199	94674	33846	94098	13
48	27228	96222	28903	95732	30570	95213	32227	94665	33874	94088	12
49	27256	96214	28931	95724	30597	95204	32254	94656	33901	94078	11
50	27284	96206	28959	95715	30625	95195	32282	94646	33929	94068	10
51	27312	96198	28987	95707	30653	95186	32309	94637	33956	94058	9
52	27340	96190	29015	95698	30680	95177	32337	94627	33983	94049	8
53	27368	96182	29042	95690	30708	95168	32364	94618	34011	94039	7
54	27396	96174	29070	95681	30736	95159	32392	94609	34038	94029	6
55	27424	96166	29098	95673	30763	95150	32419	94599	34065	94019	5
56	27452	96158	29126	95664	30791	95142	32447	94590	34093	94009	4
57	27480	96150	29154	95656	30819	95133	32474	94580	34120	93999	3
58	27508	96142	29182	95647	30846	95124	32502	94571	34147	93989	2
59	27536	96134	29209	95639	30874	95115	32529	94561	34175	93979	1
60	27564	96126	29237	95630	30902	95106	32557	94552	34202	93969	0
'	74°		73°		72°		71°		70°		'
	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	

	20°		21°		22°		23°		24°		
	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	
0	34202	93969	35837	93358	37461	92718	39073	92050	40674	91355	60
1	34229	93959	35864	93348	37488	92707	39100	92039	40700	91343	59
2	34257	93949	35891	93337	37515	92697	39127	92028	40727	91331	58
3	34284	93939	35918	93327	37542	92686	39153	92016	40753	91319	57
4	34311	93929	35945	93316	37569	92675	39180	92005	40780	91307	56
5	34339	93919	35973	93306	37595	92664	39207	91994	40806	91295	55
6	34366	93909	36000	93295	37622	92653	39234	91982	40833	91283	54
7	34393	93899	36027	93285	37649	92642	39260	91971	40860	91272	53
8	34421	93889	36054	93274	37676	92631	39287	91959	40886	91260	52
9	34448	93879	36081	93264	37703	92620	39314	91948	40913	91248	51
10	34475	93869	36108	93253	37730	92609	39341	91936	40939	91236	50
11	34503	93859	36135	93243	37757	92598	39367	91925	40966	91224	49
12	34530	93849	36162	93232	37784	92587	39394	91914	40992	91212	48
13	34557	93839	36190	93222	37811	92576	39421	91902	41019	91200	47
14	34584	93829	36217	93211	37838	92565	39448	91891	41045	91188	46
15	34612	93819	36244	93201	37865	92554	39474	91879	41072	91176	45
16	34639	93809	36271	93190	37892	92543	39501	91868	41098	91164	44
17	34666	93799	36298	93180	37919	92532	39528	91856	41125	91152	43
18	34694	93789	36325	93169	37946	92521	39555	91845	41151	91140	42
19	34721	93779	36352	93159	37973	92510	39581	91833	41178	91128	41
20	34748	93769	36379	93148	37999	92499	39608	91822	41204	91116	40
21	34775	93759	36406	93137	38026	92488	39635	91810	41231	91104	39
22	34803	93748	36434	93127	38053	92477	39661	91799	41257	91092	38
23	34830	93738	36461	93116	38080	92466	39688	91787	41284	91080	37
24	34857	93728	36488	93106	38107	92455	39715	91775	41310	91068	36
25	34884	93718	36515	93095	38134	92444	39741	91764	41337	91056	35
26	34912	93708	36542	93084	38161	92432	39768	91752	41363	91044	34
27	34939	93698	36569	93074	38188	92421	39795	91741	41390	91032	33
28	34966	93688	36596	93063	38215	92410	39822	91729	41416	91020	32
29	34993	93677	36623	93052	38241	92399	39848	91718	41443	91008	31
30	35021	93667	36650	93042	38268	92388	39875	91706	41469	90996	30
31	35048	93657	36677	93031	38295	92377	39902	91694	41496	90984	29
32	35075	93647	36704	93020	38322	92366	39928	91683	41522	90972	28
33	35102	93637	36731	93010	38349	92355	39955	91671	41549	90960	27
34	35130	93626	36758	92999	38376	92343	39982	91660	41575	90948	26
35	35157	93616	36785	92988	38403	92332	40008	91648	41602	90936	25
36	35184	93606	36812	92978	38430	92321	40035	91636	41628	90924	24
37	35211	93596	36839	92967	38456	92310	40062	91625	41655	90911	23
38	35239	93585	36867	92956	38483	92299	40088	91613	41681	90899	22
39	35266	93575	36894	92945	38510	92287	40115	91601	41707	90887	21
40	35293	93565	36921	92935	38537	92276	40141	91590	41734	90875	20
41	35320	93555	36948	92924	38564	92265	40168	91578	41760	90863	19
42	35347	93544	36975	92913	38591	92254	40195	91566	41787	90851	18
43	35375	93534	37002	92902	38617	92243	40221	91555	41813	90839	17
44	35402	93524	37029	92892	38644	92231	40248	91543	41840	90826	16
45	35429	93514	37056	92881	38671	92220	40275	91531	41866	90814	15
46	35456	93503	37083	92870	38698	92209	40301	91519	41892	90802	14
47	35484	93493	37110	92859	38725	92198	40328	91508	41919	90790	13
48	35511	93483	37137	92849	38752	92186	40355	91496	41945	90778	12
49	35538	93472	37164	92838	38778	92175	40381	91484	41972	90766	11
50	35565	93462	37191	92827	38805	92164	40408	91472	41998	90753	10
51	35592	93452	37218	92816	38832	92152	40434	91461	42024	90741	9
52	35619	93441	37245	92805	38859	92141	40461	91449	42051	90729	8
53	35647	93431	37272	92794	38886	92130	40488	91437	42077	90717	7
54	35674	93420	37299	92784	38912	92119	40514	91425	42104	90704	6
55	35701	93410	37326	92773	38939	92107	40541	91414	42130	90692	5
56	35728	93400	37353	92762	38966	92096	40567	91402	42156	90680	4
57	35755	93389	37380	92751	38993	92085	40594	91390	42183	90668	3
58	35782	93379	37407	92740	39020	92073	40621	91378	42209	90655	2
59	35810	93368	37434	92729	39046	92062	40647	91366	42235	90643	1
60	35837	93358	37461	92718	39073	92050	40674	91355	42262	90631	0
	69°		68°		67°		66°		65°		
	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	

TABLE III.

NATURAL SINES AND COSINES.

69

°	25°		26°		27°		28°		29°		°
	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	
0	42262	90631	43837	89879	45399	89101	46947	88295	48481	87462	60
1	42288	90618	43863	89867	45425	89087	46973	88281	48506	87448	59
2	42315	90606	43889	89854	45451	89074	46999	88267	48532	87434	58
3	42341	90594	43916	89841	45477	89061	47024	88254	48557	87420	57
4	42367	90582	43942	89828	45503	89048	47050	88240	48583	87406	56
5	42394	90569	43968	89816	45529	89035	47076	88226	48608	87391	55
6	42420	90557	43994	89803	45554	89021	47101	88213	48634	87377	54
7	42446	90545	44020	89790	45580	89008	47127	88199	48659	87363	53
8	42473	90532	44046	89777	45606	88995	47153	88185	48684	87349	52
9	42499	90520	44072	89764	45632	88981	47178	88172	48710	87335	51
10	42523	90507	44098	89752	45658	88968	47204	88158	48735	87321	50
11	42552	90495	44124	89739	45684	88955	47229	88144	48761	87306	49
12	42578	90483	44151	89726	45710	88942	47255	88130	48786	87292	48
13	42604	90470	44177	89713	45736	88928	47281	88117	48811	87278	47
14	42631	90458	44203	89700	45762	88915	47306	88103	48837	87264	46
15	42657	90446	44229	89687	45787	88902	47332	88089	48862	87250	45
16	42683	90433	44255	89674	45813	88888	47358	88075	48888	87235	44
17	42709	90421	44281	89662	45839	88875	47383	88062	48913	87221	43
18	42736	90408	44307	89649	45865	88862	47409	88048	48938	87207	42
19	42762	90396	44333	89636	45891	88848	47434	88034	48964	87193	41
20	42788	90383	44359	89623	45917	88835	47460	88020	48989	87178	40
21	42815	90371	44385	89610	45942	88822	47486	88006	49014	87164	39
22	42841	90358	44411	89597	45968	88808	47511	87993	49040	87150	38
23	42867	90346	44437	89584	45994	88795	47537	87979	49065	87136	37
24	42894	90334	44464	89571	46020	88782	47562	87965	49090	87121	36
25	42920	90321	44490	89558	46046	88768	47588	87951	49116	87107	35
26	42946	90309	44516	89545	46072	88755	47614	87937	49141	87093	34
27	42972	90296	44542	89532	46097	88741	47639	87923	49166	87079	33
28	42999	90284	44568	89519	46123	88728	47665	87909	49192	87064	32
29	43023	90271	44594	89506	46149	88715	47690	87896	49217	87050	31
30	43051	90259	44620	89493	46175	88701	47716	87882	49242	87036	30
31	43077	90246	44646	89480	46201	88688	47741	87868	49268	87021	29
32	43104	90233	44672	89467	46226	88674	47767	87854	49293	87007	28
33	43130	90221	44698	89454	46252	88661	47793	87840	49318	86993	27
34	43156	90208	44724	89441	46278	88647	47818	87826	49344	86978	26
35	43182	90196	44750	89428	46304	88634	47844	87812	49369	86964	25
36	43209	90183	44776	89415	46330	88620	47869	87798	49394	86949	24
37	43235	90171	44802	89402	46355	88607	47895	87784	49419	86935	23
38	43261	90158	44828	89389	46381	88593	47920	87770	49445	86921	22
39	43287	90146	44854	89376	46407	88580	47946	87756	49470	86906	21
40	43313	90133	44880	89363	46433	88566	47971	87743	49495	86892	20
41	43340	90120	44906	89350	46458	88553	47997	87729	49521	86878	19
42	43366	90108	44932	89337	46484	88539	48022	87715	49546	86863	18
43	43392	90095	44958	89324	46510	88526	48048	87701	49571	86849	17
44	43418	90082	44984	89311	46536	88512	48073	87687	49596	86834	16
45	43445	90070	45010	89298	46561	88499	48099	87673	49622	86820	15
46	43471	90057	45036	89285	46587	88485	48124	87659	49647	86805	14
47	43497	90045	45062	89272	46613	88472	48150	87645	49672	86791	13
48	43523	90032	45088	89259	46639	88458	48175	87631	49697	86777	12
49	43549	90019	45114	89245	46664	88445	48201	87617	49723	86762	11
50	43575	90007	45140	89232	46690	88431	48226	87603	49748	86748	10
51	43602	89994	45166	89219	46716	88417	48252	87589	49773	86733	9
52	43628	89981	45192	89206	46742	88404	48277	87575	49798	86719	8
53	43654	89968	45218	89193	46767	88390	48303	87561	49824	86704	7
54	43680	89956	45243	89180	46793	88377	48328	87546	49849	86690	6
55	43706	89943	45269	89167	46819	88363	48354	87532	49874	86675	5
56	43733	89930	45295	89153	46844	88349	48379	87518	49899	86661	4
57	43759	89918	45321	89140	46870	88336	48405	87504	49924	86646	3
58	43785	89905	45347	89127	46896	88322	48430	87490	49950	86632	2
59	43811	89892	45373	89114	46921	88308	48456	87476	49975	86617	1
60	43837	89879	45399	89101	46947	88295	48481	87462	50000	86603	0
°	25°		26°		27°		28°		29°		°
	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	
64°											
68°											
62°											
61°											
60°											

°	30°		31°		32°		33°		34°		°
	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	
0	50000	86603	51504	85717	52992	84805	54464	83867	55919	82904	60
1	50025	86588	51529	85702	53017	84789	54488	83851	55943	82887	59
2	50050	86573	51554	85687	53041	84774	54513	83835	55968	82871	58
3	50076	86559	51579	85672	53066	84759	54537	83819	55992	82855	57
4	50101	86544	51604	85657	53091	84743	54561	83804	56016	82839	56
5	50126	86530	51628	85642	53115	84728	54586	83788	56040	82822	55
6	50151	86515	51653	85627	53140	84712	54610	83772	56064	82806	54
7	50176	86501	51678	85612	53164	84697	54635	83756	56088	82790	53
8	50201	86486	51703	85597	53189	84681	54659	83740	56112	82773	52
9	50227	86471	51728	85582	53214	84666	54683	83724	56136	82757	51
10	50252	86457	51753	85567	53238	84650	54708	83708	56160	82741	50
11	50277	86442	51778	85551	53263	84635	54732	83692	56184	82724	49
12	50302	86427	51803	85536	53288	84619	54756	83676	56208	82708	48
13	50327	86413	51828	85521	53312	84604	54781	83660	56232	82692	47
14	50352	86398	51852	85506	53337	84588	54805	83645	56256	82675	46
15	50377	86384	51877	85491	53361	84573	54829	83629	56280	82659	45
16	50403	86369	51902	85476	53386	84557	54854	83613	56305	82643	44
17	50428	86354	51927	85461	53411	84542	54878	83597	56329	82626	43
18	50453	86340	51952	85446	53435	84526	54902	83581	56353	82610	42
19	50478	86325	51977	85431	53460	84511	54927	83565	56377	82593	41
20	50503	86310	52002	85416	53484	84495	54951	83549	56401	82577	40
21	50528	86295	52026	85401	53509	84480	54975	83533	56425	82561	39
22	50553	86281	52051	85385	53534	84464	54999	83517	56449	82544	38
23	50578	86266	52076	85370	53558	84448	55024	83501	56473	82528	37
24	50603	86251	52101	85355	53583	84433	55048	83485	56497	82511	36
25	50628	86237	52126	85340	53607	84417	55072	83469	56521	82495	35
26	50654	86222	52151	85325	53632	84402	55097	83453	56545	82478	34
27	50679	86207	52175	85310	53656	84386	55121	83437	56569	82462	33
28	50704	86192	52200	85294	53681	84370	55145	83421	56593	82446	32
29	50729	86178	52225	85279	53705	84355	55169	83405	56617	82429	31
30	50754	86163	52250	85264	53730	84339	55194	83389	56641	82413	30
31	50779	86148	52275	85249	53754	84324	55218	83373	56665	82396	29
32	50804	86133	52299	85234	53779	84308	55242	83356	56689	82380	28
33	50829	86119	52324	85218	53804	84292	55266	83340	56713	82363	27
34	50854	86104	52349	85203	53828	84277	55291	83324	56736	82347	26
35	50879	86089	52374	85188	53853	84261	55315	83308	56760	82330	25
36	50904	86074	52399	85173	53877	84245	55339	83292	56784	82314	24
37	50929	86059	52423	85157	53902	84230	55363	83276	56808	82297	23
38	50954	86045	52448	85142	53926	84214	55388	83260	56832	82281	22
39	50979	86030	52473	85127	53951	84198	55412	83244	56856	82264	21
40	51004	86015	52498	85112	53975	84182	55436	83228	56880	82248	20
41	51029	86000	52522	85096	54000	84167	55460	83212	56904	82231	19
42	51054	85985	52547	85081	54024	84151	55484	83195	56928	82214	18
43	51079	85970	52572	85066	54049	84135	55509	83179	56952	82198	17
44	51104	85956	52597	85051	54073	84120	55533	83163	56976	82181	16
45	51129	85941	52621	85035	54097	84104	55557	83147	57000	82165	15
46	51154	85926	52646	85020	54122	84088	55581	83131	57024	82148	14
47	51179	85911	52671	85005	54146	84072	55605	83115	57047	82132	13
48	51204	85896	52696	84989	54171	84057	55630	83098	57071	82115	12
49	51229	85881	52720	84974	54195	84041	55654	83082	57095	82098	11
50	51254	85866	52745	84959	54220	84025	55678	83066	57119	82082	10
51	51279	85851	52770	84943	54244	84009	55702	83050	57143	82065	9
52	51304	85836	52794	84928	54269	83994	55726	83034	57167	82048	8
53	51329	85821	52819	84913	54293	83978	55750	83017	57191	82032	7
54	51354	85806	52844	84897	54317	83962	55775	83001	57215	82015	6
55	51379	85792	52869	84882	54342	83946	55799	82985	57238	81999	5
56	51404	85777	52893	84866	54366	83930	55823	82969	57262	81982	4
57	51429	85762	52918	84851	54391	83915	55847	82953	57286	81965	3
58	51454	85747	52943	84836	54415	83899	55871	82936	57310	81949	2
59	51479	85732	52967	84820	54440	83883	55895	82920	57334	81932	1
60	51504	85717	52992	84805	54464	83867	55919	82904	57358	81915	0
°	59°		58°		57°		56°		55°		°
	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	

TABLE III. NATURAL SINES AND COSINES.												71
°	35°		36°		37°		38°		39°		'	
	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.		
0	57358	81915	58779	80902	60182	79864	61566	78801	62932	77715	60	
1	57381	81899	58802	80885	60205	79846	61589	78783	62955	77696	59	
2	57405	81882	58826	80867	60228	79829	61612	78765	62977	77678	58	
3	57429	81865	58849	80850	60251	79811	61635	78747	63000	77660	57	
4	57453	81848	58873	80833	60274	79793	61658	78729	63022	77641	56	
5	57477	81832	58896	80816	60298	79776	61681	78711	63045	77623	55	
6	57501	81815	58920	80799	60321	79758	61704	78694	63068	77605	54	
7	57524	81798	58943	80782	60344	79741	61726	78676	63090	77586	53	
8	57548	81782	58967	80765	60367	79723	61749	78658	63113	77568	52	
9	57572	81765	58990	80748	60390	79706	61772	78640	63135	77550	51	
10	57596	81748	59014	80730	60414	79688	61795	78622	63158	77531	50	
11	57619	81731	59037	80713	60437	79671	61818	78604	63180	77513	49	
12	57643	81714	59061	80696	60460	79653	61841	78586	63203	77494	48	
13	57667	81698	59084	80679	60483	79635	61864	78568	63225	77476	47	
14	57691	81681	59108	80662	60506	79618	61887	78550	63248	77458	46	
15	57715	81664	59131	80644	60529	79600	61909	78532	63271	77439	45	
16	57738	81647	59154	80627	60553	79583	61932	78514	63293	77421	44	
17	57762	81631	59178	80610	60576	79565	61955	78496	63316	77402	43	
18	57786	81614	59201	80593	60599	79547	61978	78478	63338	77384	42	
19	57810	81597	59225	80576	60622	79530	62001	78460	63361	77366	41	
20	57833	81580	59248	80558	60645	79512	62024	78442	63383	77347	40	
21	57857	81563	59272	80541	60668	79494	62046	78424	63406	77329	39	
22	57881	81546	59295	80524	60691	79477	62069	78405	63428	77310	38	
23	57904	81530	59318	80507	60714	79459	62092	78387	63451	77292	37	
24	57928	81513	59342	80489	60738	79441	62115	78369	63473	77273	36	
25	57952	81496	59365	80472	60761	79424	62138	78351	63496	77255	35	
26	57976	81479	59389	80455	60784	79406	62160	78333	63518	77236	34	
27	57999	81462	59412	80438	60807	79388	62183	78315	63540	77218	33	
28	58023	81445	59436	80420	60830	79371	62206	78297	63562	77199	32	
29	58047	81428	59459	80403	60853	79353	62229	78279	63585	77181	31	
30	58070	81412	59482	80386	60876	79335	62251	78261	63608	77162	30	
31	58094	81395	59506	80368	60899	79318	62274	78243	63630	77144	29	
32	58118	81378	59529	80351	60922	79300	62297	78225	63653	77125	28	
33	58141	81361	59552	80334	60945	79282	62320	78206	63675	77107	27	
34	58165	81344	59576	80316	60968	79264	62342	78188	63698	77088	26	
35	58189	81327	59599	80299	60991	79247	62365	78170	63720	77070	25	
36	58212	81310	59622	80282	61015	79229	62388	78152	63742	77051	24	
37	58236	81293	59646	80264	61038	79211	62411	78134	63765	77033	23	
38	58260	81276	59669	80247	61061	79193	62433	78116	63787	77014	22	
39	58283	81259	59693	80230	61084	79176	62456	78098	63810	76996	21	
40	58307	81242	59716	80212	61107	79158	62479	78079	63832	76977	20	
41	58330	81225	59739	80195	61130	79140	62502	78061	63854	76959	19	
42	58354	81208	59763	80178	61153	79122	62524	78043	63877	76940	18	
43	58378	81191	59786	80160	61176	79105	62547	78025	63899	76921	17	
44	58401	81174	59809	80143	61199	79087	62570	78007	63922	76903	16	
45	58425	81157	59832	80125	61222	79069	62592	77988	63944	76884	15	
46	58449	81140	59856	80108	61245	79051	62615	77970	63966	76866	14	
47	58472	81123	59879	80091	61268	79033	62638	77952	63989	76847	13	
48	58496	81106	59902	80073	61291	79016	62660	77934	64011	76828	12	
49	58519	81089	59926	80056	61314	78998	62683	77916	64033	76810	11	
50	58543	81072	59949	80038	61337	78980	62706	77897	64056	76791	10	
51	58567	81055	59972	80021	61360	78962	62728	77879	64078	76772	9	
52	58590	81038	59995	80003	61383	78944	62751	77861	64100	76754	8	
53	58614	81021	60019	79986	61406	78926	62774	77843	64123	76735	7	
54	58637	81004	60042	79968	61429	78908	62796	77824	64145	76717	6	
55	58661	80987	60065	79951	61451	78891	62819	77806	64167	76698	5	
56	58684	80970	60089	79934	61474	78873	62842	77788	64190	76679	4	
57	58708	80953	60112	79916	61497	78855	62864	77769	64212	76661	3	
58	58731	80936	60135	79899	61520	78837	62887	77751	64234	76642	2	
59	58755	80919	60158	79881	61543	78819	62909	77733	64256	76623	1	
60	58779	80902	60182	79864	61566	78801	62932	77715	64279	76604	0	
'	Cosine.		Sine.		Cosine.		Sine.		Cosine.		'	
	54°		53°		52°		51°		50°			

°	40°		41°		42°		43°		44°		°
	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	
0	64279	76604	65606	75471	66913	74314	68200	73135	69466	71934	60
1	64301	76586	65628	75452	66935	74295	68221	73116	69487	71914	59
2	64323	76567	65650	75433	66956	74276	68242	73096	69508	71894	58
3	64346	76548	65672	75414	66978	74256	68264	73076	69529	71873	57
4	64368	76530	65694	75395	66999	74237	68285	73056	69549	71853	56
5	64390	76511	65716	75375	67021	74217	68306	73036	69570	71833	55
6	64412	76492	65738	75356	67043	74198	68327	73016	69591	71813	54
7	64435	76473	65759	75337	67064	74178	68349	72996	69612	71792	53
8	64457	76455	65781	75318	67086	74159	68370	72976	69633	71772	52
9	64479	76436	65803	75299	67107	74139	68391	72957	69654	71752	51
10	64501	76417	65825	75280	67129	74120	68412	72937	69675	71732	50
11	64524	76398	65847	75261	67151	74100	68434	72917	69696	71711	49
12	64546	76380	65869	75241	67172	74080	68455	72897	69717	71691	48
13	64568	76361	65891	75222	67194	74061	68476	72877	69737	71671	47
14	64590	76342	65913	75203	67215	74041	68497	72857	69758	71650	46
15	64612	76323	65935	75184	67237	74022	68518	72837	69779	71630	45
16	64635	76304	65956	75165	67258	74002	68539	72817	69800	71610	44
17	64657	76286	65978	75146	67280	73983	68561	72797	69821	71590	43
18	64679	76267	66000	75126	67301	73963	68582	72777	69842	71569	42
19	64701	76248	66022	75107	67323	73944	68603	72757	69862	71549	41
20	64723	76229	66044	75088	67344	73924	68624	72737	69883	71529	40
21	64746	76210	66066	75069	67366	73904	68645	72717	69904	71508	39
22	64768	76192	66088	75050	67387	73885	68666	72697	69925	71488	38
23	64790	76173	66109	75030	67409	73865	68688	72677	69946	71468	37
24	64812	76154	66131	75011	67430	73846	68709	72657	69966	71447	36
25	64834	76135	66153	74992	67452	73826	68730	72637	69987	71427	35
26	64856	76116	66175	74973	67473	73806	68751	72617	70008	71407	34
27	64878	76097	66197	74953	67495	73787	68772	72597	70029	71386	33
28	64901	76078	66218	74934	67516	73767	68793	72577	70049	71366	32
29	64923	76059	66240	74915	67538	73747	68814	72557	70070	71345	31
30	64945	76041	66262	74896	67559	73728	68835	72537	70091	71325	30
31	64967	76022	66284	74876	67580	73708	68857	72517	70112	71305	29
32	64989	76003	66306	74857	67602	73688	68878	72497	70132	71284	28
33	65011	75984	66327	74838	67623	73669	68899	72477	70153	71264	27
34	65033	75965	66349	74818	67645	73649	68920	72457	70174	71243	26
35	65055	75946	66371	74799	67666	73629	68941	72437	70195	71223	25
36	65077	75927	66393	74780	67688	73610	68962	72417	70215	71203	24
37	65100	75908	66414	74760	67709	73590	68983	72397	70236	71182	23
38	65122	75889	66436	74741	67730	73570	69004	72377	70257	71162	22
39	65144	75870	66458	74722	67752	73551	69025	72357	70277	71141	21
40	65166	75851	66480	74703	67773	73531	69046	72337	70298	71121	20
41	65188	75832	66501	74683	67795	73511	69067	72317	70319	71100	19
42	65210	75813	66523	74664	67816	73491	69088	72297	70339	71080	18
43	65232	75794	66545	74644	67837	73472	69109	72277	70360	71059	17
44	65254	75775	66566	74625	67859	73452	69130	72257	70381	71039	16
45	65276	75756	66588	74606	67880	73432	69151	72236	70401	71019	15
46	65298	75738	66610	74586	67901	73413	69172	72216	70422	70998	14
47	65320	75719	66632	74567	67923	73393	69193	72196	70443	70978	13
48	65342	75700	66653	74548	67944	73373	69214	72176	70463	70957	12
49	65364	75680	66675	74528	67965	73353	69235	72156	70484	70937	11
50	65386	75661	66697	74509	67987	73333	69256	72136	70505	70916	10
51	65408	75642	66718	74489	68008	73314	69277	72116	70525	70896	9
52	65430	75623	66740	74470	68029	73294	69298	72095	70546	70875	8
53	65452	75604	66762	74451	68051	73274	69319	72075	70567	70855	7
54	65474	75585	66783	74431	68072	73254	69340	72055	70587	70834	6
55	65496	75566	66805	74412	68093	73234	69361	72035	70608	70813	5
56	65518	75547	66827	74392	68115	73215	69382	72015	70628	70793	4
57	65540	75528	66848	74373	68136	73195	69403	71995	70649	70772	3
58	65562	75509	66870	74353	68157	73175	69424	71974	70670	70752	2
59	65584	75490	66891	74334	68179	73155	69445	71954	70690	70731	1
60	65606	75471	66913	74314	68200	73135	69466	71934	70711	70711	0
°	49°		48°		47°		46°		45°		°
	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	

TABLE III. NATURAL TANGENTS AND COTANGENTS.

73

°	0°		1°		2°		3°		°
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	00000	Infinite.	01746	57.2900	03492	28.6363	05241	19.0811	60
1	00029	3437.75	01775	56.3506	03521	28.3994	05270	18.9755	59
2	00058	1718.87	01804	55.4415	03550	28.1664	05299	18.8711	58
3	00087	1145.92	01833	54.5613	03579	27.9372	05328	18.7678	57
4	00116	859.436	01862	53.7086	03609	27.7117	05357	18.6656	56
5	00145	687.549	01891	52.8821	03638	27.4899	05387	18.5645	55
6	00175	572.957	01920	52.0807	03667	27.2715	05416	18.4645	54
7	00204	491.106	01949	51.3032	03696	27.0566	05445	18.3655	53
8	00233	429.718	01978	50.5485	03725	26.8450	05474	18.2677	52
9	00262	381.971	02007	49.8157	03754	26.6367	05503	18.1708	51
10	00291	343.774	02036	49.1039	03783	26.4316	05532	18.0750	50
11	00320	312.521	02066	48.4121	03812	26.2296	05562	17.9802	49
12	00349	286.478	02095	47.7395	03842	26.0307	05591	17.8863	48
13	00378	264.441	02124	47.0853	03871	25.8348	05620	17.7934	47
14	00407	245.552	02153	46.4489	03900	25.6418	05649	17.7015	46
15	00436	229.182	02182	45.8294	03929	25.4517	05678	17.6106	45
16	00465	214.858	02211	45.2261	03958	25.2644	05708	17.5205	44
17	00495	202.219	02240	44.6386	03987	25.0798	05737	17.4314	43
18	00524	190.984	02269	44.0661	04016	24.8978	05766	17.3432	42
19	00553	180.932	02298	43.5081	04046	24.7185	05795	17.2558	41
20	00582	171.885	02328	42.9641	04075	24.5418	05824	17.1693	40
21	00611	163.700	02357	42.4335	04104	24.3675	05854	17.0837	39
22	00640	156.259	02386	41.9158	04133	24.1957	05883	16.9990	38
23	00669	149.465	02415	41.4106	04162	24.0263	05912	16.9150	37
24	00698	143.237	02444	40.9174	04191	23.8593	05941	16.8319	36
25	00727	137.507	02473	40.4358	04220	23.6945	05970	16.7496	35
26	00756	132.219	02502	39.9655	04250	23.5321	05999	16.6681	34
27	00785	127.321	02531	39.5059	04279	23.3718	06029	16.5874	33
28	00814	122.774	02560	39.0568	04308	23.2137	06058	16.5075	32
29	00844	118.540	02589	38.6177	04337	23.0577	06087	16.4283	31
30	00873	114.589	02619	38.1885	04366	22.9038	06116	16.3499	30
31	00902	110.892	02648	37.7686	04395	22.7519	06145	16.2722	29
32	00931	107.426	02677	37.3579	04424	22.6020	06175	16.1952	28
33	00960	104.171	02706	36.9560	04454	22.4541	06204	16.1190	27
34	00989	101.107	02735	36.5627	04483	22.3081	06233	16.0435	26
35	01018	98.2179	02764	36.1776	04512	22.1640	06262	15.9687	25
36	01047	95.4895	02793	35.8006	04541	22.0217	06291	15.8945	24
37	01076	92.9085	02822	35.4313	04570	21.8813	06321	15.8211	23
38	01105	90.4633	02851	35.0695	04599	21.7426	06350	15.7483	22
39	01135	88.1436	02881	34.7151	04628	21.6056	06379	15.6762	21
40	01164	85.9398	02910	34.3678	04658	21.4704	06408	15.6048	20
41	01193	83.8435	02939	34.0273	04687	21.3369	06437	15.5340	19
42	01222	81.8470	02968	33.6935	04716	21.2049	06467	15.4638	18
43	01251	79.9434	02997	33.3662	04745	21.0747	06496	15.3943	17
44	01280	78.1263	03026	33.0452	04774	20.9460	06525	15.3254	16
45	01309	76.3900	03055	32.7303	04803	20.8188	06554	15.2571	15
46	01338	74.7292	03084	32.4213	04832	20.6932	06584	15.1893	14
47	01367	73.1390	03114	32.1181	04862	20.5691	06613	15.1222	13
48	01396	71.6151	03143	31.8205	04891	20.4465	06642	15.0557	12
49	01425	70.1533	03172	31.5284	04920	20.3253	06671	14.9898	11
50	01455	68.7501	03201	31.2416	04949	20.2056	06700	14.9244	10
51	01484	67.4019	03230	30.9599	04978	20.0872	06730	14.8596	9
52	01513	66.1055	03259	30.6833	05007	19.9702	06759	14.7954	8
53	01542	64.8580	03288	30.4116	05037	19.8546	06788	14.7317	7
54	01571	63.6567	03317	30.1446	05066	19.7403	06817	14.6685	6
55	01600	62.4992	03346	29.8823	05095	19.6273	06847	14.6059	5
56	01629	61.3829	03376	29.6245	05124	19.5156	06876	14.5438	4
57	01658	60.3058	03405	29.3711	05153	19.4051	06905	14.4823	3
58	01687	59.2659	03434	29.1220	05182	19.2959	06934	14.4212	2
59	01716	58.2612	03463	28.8771	05212	19.1879	06963	14.3607	1
60	01746	57.2900	03492	28.6363	05241	19.0811	06993	14.3007	0
Cotang.		Tangent.	Cotang.		Tangent.	Cotang.		Tangent.	°
89°			88°			87°		86°	

°	4°		5°		6°		7°		°
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	06993	14.3007	08749	11.4301	10510	9.51436	12278	8.14435	60
1	07022	14.2411	08778	11.3919	10540	9.48781	12308	8.12481	59
2	07051	14.1821	08807	11.3540	10569	9.46141	12338	8.10536	58
3	07080	14.1235	08837	11.3163	10599	9.43515	12367	8.08600	57
4	07110	14.0655	08866	11.2789	10628	9.40904	12397	8.06674	56
5	07139	14.0079	08895	11.2417	10657	9.38307	12426	8.04756	55
6	07168	13.9507	08925	11.2048	10687	9.35724	12456	8.02848	54
7	07197	13.8940	08954	11.1681	10716	9.33154	12485	8.00948	53
8	07227	13.8378	08983	11.1316	10746	9.30599	12515	7.99058	52
9	07256	13.7821	09013	11.0954	10775	9.28058	12544	7.97176	51
10	07285	13.7267	09042	11.0594	10805	9.25530	12574	7.95302	50
11	07314	13.6719	09071	11.0237	10834	9.23016	12603	7.93438	49
12	07344	13.6174	09101	10.9882	10863	9.20516	12633	7.91582	48
13	07373	13.5634	09130	10.9529	10893	9.18028	12662	7.89734	47
14	07402	13.5098	09159	10.9178	10922	9.15554	12692	7.87895	46
15	07431	13.4566	09189	10.8829	10952	9.13093	12722	7.86064	45
16	07461	13.4039	09218	10.8483	10981	9.10646	12751	7.84242	44
17	07490	13.3515	09247	10.8139	11011	9.08211	12781	7.82428	43
18	07519	13.2996	09277	10.7797	11040	9.05789	12810	7.80622	42
19	07548	13.2480	09306	10.7457	11070	9.03379	12840	7.78825	41
20	07578	13.1969	09335	10.7119	11099	9.00983	12869	7.77035	40
21	07607	13.1461	09365	10.6783	11128	8.98598	12899	7.75254	39
22	07636	13.0958	09394	10.6450	11158	8.96227	12929	7.73480	38
23	07665	13.0458	09423	10.6118	11187	8.93867	12958	7.71715	37
24	07695	12.9962	09453	10.5789	11217	8.91520	12988	7.69957	36
25	07724	12.9469	09482	10.5462	11246	8.89185	13017	7.68208	35
26	07753	12.8981	09511	10.5136	11276	8.86862	13047	7.66466	34
27	07782	12.8496	09541	10.4813	11305	8.84551	13076	7.64732	33
28	07812	12.8014	09570	10.4491	11335	8.82252	13106	7.63005	32
29	07841	12.7536	09600	10.4172	11364	8.79964	13136	7.61287	31
30	07870	12.7062	09629	10.3854	11394	8.77689	13165	7.59575	30
31	07899	12.6591	09658	10.3538	11423	8.75425	13195	7.57872	29
32	07929	12.6124	09688	10.3224	11452	8.73172	13224	7.56176	28
33	07958	12.5660	09717	10.2913	11482	8.70931	13254	7.54487	27
34	07987	12.5199	09746	10.2602	11511	8.68701	13284	7.52806	26
35	08017	12.4742	09776	10.2294	11541	8.66482	13313	7.51132	25
36	08046	12.4288	09805	10.1988	11570	8.64275	13343	7.49465	24
37	08075	12.3838	09834	10.1683	11600	8.62078	13372	7.47806	23
38	08104	12.3390	09864	10.1381	11629	8.59893	13402	7.46154	22
39	08134	12.2946	09893	10.1080	11659	8.57718	13432	7.44509	21
40	08163	12.2505	09923	10.0780	11688	8.55555	13461	7.42871	20
41	08192	12.2067	09952	10.0483	11718	8.53402	13491	7.41240	19
42	08221	12.1632	09981	10.0187	11747	8.51259	13521	7.39616	18
43	08251	12.1201	10011	9.98930	11777	8.49128	13550	7.37999	17
44	08280	12.0772	10040	9.96007	11806	8.47007	13580	7.36389	16
45	08309	12.0346	10069	9.93101	11836	8.44896	13609	7.34786	15
46	08339	11.9923	10099	9.90211	11865	8.42795	13639	7.33190	14
47	08368	11.9504	10128	9.87338	11895	8.40705	13669	7.31600	13
48	08397	11.9087	10158	9.84482	11924	8.38625	13698	7.30018	12
49	08427	11.8673	10187	9.81641	11954	8.36555	13728	7.28442	11
50	08456	11.8262	10216	9.78817	11983	8.34496	13758	7.26873	10
51	08485	11.7853	10246	9.76009	12013	8.32446	13787	7.25310	9
52	08514	11.7448	10275	9.73217	12042	8.30406	13817	7.23754	8
53	08544	11.7045	10305	9.70441	12072	8.28376	13846	7.22204	7
54	08573	11.6645	10334	9.67680	12101	8.26355	13876	7.20661	6
55	08602	11.6248	10363	9.64935	12131	8.24345	13906	7.19125	5
56	08632	11.5853	10393	9.62205	12160	8.22344	13935	7.17594	4
57	08661	11.5461	10422	9.59490	12190	8.20352	13965	7.16071	3
58	08690	11.5072	10452	9.56791	12219	8.18370	13995	7.14553	2
59	08720	11.4685	10481	9.54106	12249	8.16398	14024	7.13042	1
60	08749	11.4301	10510	9.51436	12278	8.14435	14054	7.11537	0
°	85°		84°		83°		82°		°
	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	

TABLE III. NATURAL TANGENTS AND COTANGENTS. 75

°	8°		9°		10°		11°		°
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	14054	7.11537	15838	6.31375	17633	5.67128	19438	5.14455	60
1	14084	7.10038	15868	6.30189	17663	5.66165	19468	5.13658	59
2	14113	7.08546	15898	6.29007	17693	5.65205	19498	5.12862	58
3	14143	7.07059	15928	6.27829	17723	5.64248	19529	5.12069	57
4	14173	7.05579	15958	6.26655	17753	5.63295	19559	5.11279	56
5	14202	7.04105	15988	6.25486	17783	5.62344	19589	5.10490	55
6	14232	7.02637	16017	6.24321	17813	5.61397	19619	5.09704	54
7	14262	7.01174	16047	6.23160	17843	5.60452	19649	5.08921	53
8	14291	6.99718	16077	6.22003	17873	5.59511	19680	5.08139	52
9	14321	6.98268	16107	6.20851	17903	5.58573	19710	5.07360	51
10	14351	6.96823	16137	6.19703	17933	5.57638	19740	5.06584	50
11	14381	6.95385	16167	6.18559	17963	5.56706	19770	5.05809	49
12	14410	6.93952	16196	6.17419	17993	5.55777	19801	5.05037	48
13	14440	6.92525	16226	6.16283	18023	5.54851	19831	5.04267	47
14	14470	6.91104	16256	6.15151	18053	5.53927	19861	5.03499	46
15	14499	6.89688	16286	6.14023	18083	5.53007	19891	5.02734	45
16	14529	6.88278	16316	6.12899	18113	5.52090	19921	5.01971	44
17	14559	6.86874	16346	6.11779	18143	5.51176	19952	5.01210	43
18	14588	6.85475	16376	6.10664	18173	5.50264	19982	5.00451	42
19	14618	6.84082	16405	6.09552	18203	5.49356	20012	4.99695	41
20	14648	6.82694	16435	6.08444	18233	5.48451	20042	4.98940	40
21	14678	6.81312	16465	6.07340	18263	5.47548	20073	4.98188	39
22	14707	6.79936	16495	6.06240	18293	5.46648	20103	4.97438	38
23	14737	6.78564	16525	6.05143	18323	5.45751	20133	4.96690	37
24	14767	6.77199	16555	6.04051	18353	5.44857	20164	4.95945	36
25	14796	6.75838	16585	6.02962	18383	5.43966	20194	4.95201	35
26	14826	6.74483	16615	6.01878	18414	5.43077	20224	4.94460	34
27	14856	6.73133	16645	6.00797	18444	5.42192	20254	4.93721	33
28	14886	6.71789	16674	5.99720	18474	5.41309	20285	4.92982	32
29	14915	6.70450	16704	5.98646	18504	5.40429	20315	4.92249	31
30	14945	6.69116	16734	5.97576	18534	5.39552	20345	4.91516	30
31	14975	6.67787	16764	5.96510	18564	5.38677	20376	4.90785	29
32	15005	6.66463	16794	5.95448	18594	5.37805	20406	4.90056	28
33	15034	6.65144	16824	5.94390	18624	5.36936	20436	4.89330	27
34	15064	6.63831	16854	5.93335	18654	5.36070	20466	4.88605	26
35	15094	6.62523	16884	5.92283	18684	5.35206	20497	4.87882	25
36	15124	6.61219	16914	5.91235	18714	5.34345	20527	4.87162	24
37	15153	6.59921	16944	5.90191	18745	5.33487	20557	4.86444	23
38	15183	6.58627	16974	5.89151	18775	5.32631	20588	4.85727	22
39	15213	6.57339	17004	5.88114	18805	5.31778	20618	4.85013	21
40	15243	6.56055	17033	5.87080	18835	5.30928	20648	4.84300	20
41	15272	6.54777	17063	5.86051	18865	5.30080	20679	4.83590	19
42	15302	6.53503	17093	5.85024	18895	5.29235	20709	4.82882	18
43	15332	6.52234	17123	5.84001	18925	5.28393	20739	4.82177	17
44	15362	6.50970	17153	5.82982	18955	5.27553	20770	4.81471	16
45	15391	6.49710	17183	5.81966	18986	5.26715	20800	4.80769	15
46	15421	6.48456	17213	5.80953	19016	5.25880	20830	4.80068	14
47	15451	6.47206	17243	5.79944	19046	5.25048	20861	4.79370	13
48	15481	6.45961	17273	5.78938	19076	5.24218	20891	4.78673	12
49	15511	6.44720	17303	5.77936	19106	5.23391	20921	4.77978	11
50	15540	6.43484	17333	5.76937	19136	5.22566	20952	4.77286	10
51	15570	6.42253	17363	5.75941	19166	5.21744	20982	4.76595	9
52	15600	6.41026	17393	5.74949	19197	5.20925	21013	4.75906	8
53	15630	6.39804	17423	5.73960	19227	5.20107	21043	4.75219	7
54	15660	6.38587	17453	5.72974	19257	5.19293	21073	4.74534	6
55	15689	6.37374	17483	5.71992	19287	5.18480	21104	4.73851	5
56	15719	6.36165	17513	5.71013	19317	5.17671	21134	4.73170	4
57	15749	6.34961	17543	5.70037	19347	5.16863	21164	4.72490	3
58	15779	6.33761	17573	5.69064	19378	5.16058	21195	4.71813	2
59	15809	6.32566	17603	5.68094	19408	5.15256	21225	4.71137	1
60	15838	6.31375	17633	5.67128	19438	5.14455	21256	4.70463	0
°	81°		80°		79°		78°		°
	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	

°	12°		13°		14°		15°		°
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	21256	4.70463	23087	4.33148	24933	4.01078	26795	3.73205	60
1	21286	4.69791	23117	4.32573	24964	4.00582	26826	3.72771	59
2	21316	4.69121	23148	4.32001	24995	4.00086	26857	3.72338	58
3	21347	4.68452	23179	4.31430	25026	3.99592	26888	3.71907	57
4	21377	4.67786	23209	4.30860	25056	3.99099	26920	3.71476	56
5	21408	4.67121	23240	4.30291	25087	3.98607	26951	3.71046	55
6	21438	4.66458	23271	4.29724	25118	3.98117	26982	3.70616	54
7	21469	4.65797	23301	4.29159	25149	3.97627	27013	3.70188	53
8	21499	4.65138	23332	4.28595	25180	3.97139	27044	3.69761	52
9	21529	4.64480	23363	4.28032	25211	3.96651	27076	3.69335	51
10	21560	4.63825	23393	4.27471	25242	3.96165	27107	3.68909	50
11	21590	4.63171	23424	4.26911	25273	3.95680	27138	3.68485	49
12	21621	4.62518	23455	4.26352	25304	3.95196	27169	3.68061	48
13	21651	4.61868	23485	4.25795	25335	3.94713	27201	3.67638	47
14	21682	4.61219	23516	4.25239	25366	3.94232	27232	3.67217	46
15	21712	4.60572	23547	4.24685	25397	3.93751	27263	3.66796	45
16	21743	4.59927	23578	4.24132	25428	3.93271	27294	3.66376	44
17	21773	4.59283	23608	4.23580	25459	3.92793	27326	3.65957	43
18	21804	4.58641	23639	4.23030	25490	3.92316	27357	3.65538	42
19	21834	4.58001	23670	4.22481	25521	3.91839	27388	3.65121	41
20	21864	4.57363	23700	4.21933	25552	3.91364	27419	3.64705	40
21	21895	4.56726	23731	4.21387	25583	3.90890	27451	3.64289	39
22	21925	4.56091	23762	4.20842	25614	3.90417	27482	3.63874	38
23	21956	4.55458	23793	4.20298	25645	3.89945	27513	3.63461	37
24	21986	4.54826	23823	4.19756	25676	3.89474	27545	3.63048	36
25	22017	4.54196	23854	4.19215	25707	3.89004	27576	3.62636	35
26	22047	4.53568	23885	4.18675	25738	3.88536	27607	3.62224	34
27	22078	4.52941	23916	4.18137	25769	3.88068	27638	3.61814	33
28	22108	4.52316	23946	4.17600	25800	3.87601	27670	3.61405	32
29	22139	4.51693	23977	4.17064	25831	3.87136	27701	3.60996	31
30	22169	4.51071	24008	4.16530	25862	3.86671	27732	3.60588	30
31	22200	4.50451	24039	4.15997	25893	3.86208	27764	3.60181	29
32	22231	4.49832	24069	4.15465	25924	3.85745	27795	3.59775	28
33	22261	4.49215	24100	4.14934	25955	3.85284	27826	3.59370	27
34	22292	4.48600	24131	4.14405	25986	3.84824	27858	3.58966	26
35	22322	4.47986	24162	4.13877	26017	3.84364	27889	3.58562	25
36	22353	4.47374	24193	4.13350	26048	3.83909	27920	3.58160	24
37	22383	4.46764	24223	4.12825	26079	3.83449	27952	3.57758	23
38	22414	4.46155	24254	4.12301	26110	3.82992	27983	3.57357	22
39	22444	4.45548	24285	4.11778	26141	3.82537	28015	3.56957	21
40	22475	4.44942	24316	4.11256	26172	3.82083	28046	3.56557	20
41	22505	4.44338	24347	4.10736	26203	3.81630	28077	3.56159	19
42	22536	4.43735	24377	4.10216	26235	3.81177	28109	3.55761	18
43	22567	4.43134	24408	4.09699	26266	3.80726	28140	3.55364	17
44	22597	4.42534	24439	4.09182	26297	3.80276	28172	3.54968	16
45	22628	4.41936	24470	4.08666	26328	3.79827	28203	3.54573	15
46	22658	4.41340	24501	4.08152	26359	3.79378	28234	3.54179	14
47	22689	4.40745	24532	4.07639	26390	3.78931	28266	3.53785	13
48	22719	4.40152	24562	4.07127	26421	3.78485	28297	3.53393	12
49	22750	4.39560	24593	4.06616	26452	3.78040	28329	3.53001	11
50	22781	4.38969	24624	4.06107	26483	3.77595	28360	3.52609	10
51	22811	4.38381	24655	4.05599	26515	3.77152	28391	3.52219	9
52	22842	4.37793	24686	4.05092	26546	3.76709	28423	3.51829	8
53	22872	4.37207	24717	4.04586	26577	3.76268	28454	3.51441	7
54	22903	4.36623	24747	4.04081	26608	3.75828	28486	3.51053	6
55	22934	4.36040	24778	4.03578	26639	3.75388	28517	3.50666	5
56	22964	4.35459	24809	4.03075	26670	3.74950	28549	3.50279	4
57	22995	4.34879	24840	4.02574	26701	3.74512	28580	3.49894	3
58	23026	4.34300	24871	4.02074	26733	3.74075	28612	3.49509	2
59	23056	4.33723	24902	4.01576	26764	3.73640	28643	3.49125	1
60	23087	4.33148	24933	4.01078	26795	3.73205	28675	3.48741	0
°	12°		13°		14°		15°		°
	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	
	77°		76°		75°		74°		

TABLE III. NATURAL TANGENTS AND COTANGENTS. 77

°	16°		17°		18°		19°		°
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	28675	3.48741	30573	3.27085	32492	3.07768	34433	2.90421	60
1	28706	3.48359	30605	3.26745	32524	3.07464	34465	2.90147	59
2	28738	3.47977	30637	3.26406	32556	3.07160	34498	2.89873	58
3	28769	3.47596	30669	3.26067	32588	3.06857	34530	2.89600	57
4	28800	3.47216	30700	3.25729	32621	3.06554	34563	2.89327	56
5	28832	3.46837	30732	3.25392	32653	3.06252	34596	2.89055	55
6	28864	3.46458	30764	3.25055	32685	3.05950	34628	2.88783	54
7	28895	3.46080	30796	3.24719	32717	3.05649	34661	2.88511	53
8	28927	3.45703	30828	3.24383	32749	3.05349	34693	2.88240	52
9	28958	3.45327	30860	3.24049	32782	3.05049	34726	2.87970	51
10	28990	3.44951	30891	3.23714	32814	3.04749	34758	2.87700	50
11	29021	3.44576	30923	3.23381	32846	3.04450	34791	2.87430	49
12	29053	3.44202	30955	3.23048	32878	3.04152	34824	2.87161	48
13	29084	3.43829	30987	3.22715	32911	3.03854	34856	2.86892	47
14	29116	3.43456	31019	3.22384	32943	3.03556	34889	2.86624	46
15	29147	3.43084	31051	3.22053	32975	3.03260	34922	2.86356	45
16	29179	3.42713	31083	3.21722	33007	3.02963	34954	2.86089	44
17	29210	3.42343	31115	3.21392	33040	3.02667	34987	2.85822	43
18	29242	3.41973	31147	3.21063	33072	3.02372	35019	2.85555	42
19	29274	3.41604	31178	3.20734	33104	3.02077	35052	2.85289	41
20	29305	3.41236	31210	3.20406	33136	3.01783	35085	2.85023	40
21	29337	3.40869	31242	3.20079	33169	3.01489	35117	2.84758	39
22	29368	3.40502	31274	3.19752	33201	3.01196	35150	2.84494	38
23	29400	3.40136	31306	3.19426	33233	3.00903	35183	2.84229	37
24	29432	3.39771	31338	3.19100	33266	3.00611	35216	2.83965	36
25	29463	3.39406	31370	3.18775	33298	3.00319	35248	2.83702	35
26	29495	3.39042	31402	3.18451	33330	3.00028	35281	2.83439	34
27	29526	3.38679	31434	3.18127	33363	2.99738	35314	2.83176	33
28	29558	3.38317	31466	3.17804	33395	2.99447	35346	2.82914	32
29	29590	3.37955	31498	3.17481	33427	2.99158	35379	2.82653	31
30	29621	3.37594	31530	3.17159	33460	2.98868	35412	2.82391	30
31	29653	3.37234	31562	3.16838	33492	2.98580	35445	2.82130	29
32	29685	3.36875	31594	3.16517	33524	2.98292	35477	2.81870	28
33	29716	3.36516	31626	3.16197	33557	2.98004	35510	2.81610	27
34	29748	3.36158	31658	3.15877	33589	2.97717	35543	2.81350	26
35	29780	3.35800	31690	3.15558	33621	2.97430	35576	2.81091	25
36	29811	3.35443	31722	3.15240	33654	2.97144	35608	2.80833	24
37	29843	3.35087	31754	3.14922	33686	2.96858	35641	2.80574	23
38	29875	3.34732	31786	3.14605	33718	2.96573	35674	2.80316	22
39	29906	3.34377	31818	3.14288	33751	2.96288	35707	2.80059	21
40	29938	3.34023	31850	3.13972	33783	2.96004	35740	2.79802	20
41	29970	3.33670	31882	3.13656	33816	2.95721	35772	2.79545	19
42	30001	3.33317	31914	3.13341	33848	2.95437	35805	2.79289	18
43	30033	3.32965	31946	3.13027	33881	2.95155	35838	2.79033	17
44	30065	3.32614	31978	3.12713	33913	2.94872	35871	2.78778	16
45	30097	3.32264	32010	3.12400	33945	2.94590	35904	2.78523	15
46	30128	3.31914	32042	3.12087	33978	2.94309	35937	2.78269	14
47	30160	3.31565	32074	3.11775	34010	2.94028	35969	2.78014	13
48	30192	3.31216	32106	3.11464	34043	2.93748	36002	2.77761	12
49	30224	3.30868	32139	3.11153	34075	2.93468	36035	2.77507	11
50	30255	3.30521	32171	3.10842	34108	2.93189	36068	2.77254	10
51	30287	3.30174	32203	3.10532	34140	2.92910	36101	2.77002	9
52	30319	3.29829	32235	3.10223	34173	2.92632	36134	2.76750	8
53	30351	3.29483	32267	3.09914	34205	2.92354	36167	2.76498	7
54	30382	3.29139	32299	3.09606	34238	2.92076	36199	2.76247	6
55	30414	3.28795	32331	3.09298	34270	2.91799	36232	2.75996	5
56	30446	3.28452	32363	3.08991	34303	2.91523	36265	2.75746	4
57	30478	3.28109	32396	3.08685	34335	2.91246	36298	2.75496	3
58	30509	3.27767	32428	3.08379	34368	2.90971	36331	2.75246	2
59	30541	3.27426	32460	3.08073	34400	2.90696	36364	2.74997	1
60	30573	3.27085	32492	3.07768	34433	2.90421	36397	2.74748	0
°	16°		17°		18°		19°		°
	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	
	73°		72°		71°		70°		

	20°		21°		22°		23°		
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	36397	2.74748	38386	2.60509	40403	2.47509	42447	2.35585	60
1	36430	2.74499	38420	2.60283	40436	2.47302	42482	2.35395	59
2	36463	2.74251	38453	2.60057	40470	2.47095	42516	2.35205	58
3	36496	2.74004	38487	2.59831	40504	2.46888	42551	2.35015	57
4	36529	2.73756	38520	2.59606	40538	2.46682	42585	2.34825	56
5	36562	2.73509	38553	2.59381	40572	2.46476	42619	2.34636	55
6	36595	2.73263	38587	2.59156	40606	2.46270	42654	2.34447	54
7	36628	2.73017	38620	2.58932	40640	2.46065	42688	2.34258	53
8	36661	2.72771	38654	2.58708	40674	2.45860	42722	2.34069	52
9	36694	2.72526	38687	2.58484	40707	2.45655	42757	2.33881	51
10	36727	2.72281	38721	2.58261	40741	2.45451	42791	2.33693	50
11	36760	2.72036	38754	2.58038	40775	2.45246	42826	2.33505	49
12	36793	2.71792	38787	2.57815	40809	2.45043	42860	2.33317	48
13	36826	2.71548	38821	2.57593	40843	2.44839	42894	2.33130	47
14	36859	2.71305	38854	2.57371	40877	2.44636	42929	2.32943	46
15	36892	2.71062	38888	2.57150	40911	2.44433	42963	2.32756	45
16	36925	2.70819	38921	2.56928	40945	2.44230	42998	2.32570	44
17	36958	2.70577	38955	2.56707	40979	2.44027	43032	2.32383	43
18	36991	2.70335	38988	2.56487	41013	2.43825	43067	2.32197	42
19	37024	2.70094	39022	2.56266	41047	2.43623	43101	2.32012	41
20	37057	2.69853	39055	2.56046	41081	2.43422	43136	2.31826	40
21	37090	2.69612	39089	2.55827	41115	2.43220	43170	2.31641	39
22	37124	2.69371	39122	2.55608	41149	2.43019	43205	2.31456	38
23	37157	2.69131	39156	2.55389	41183	2.42819	43239	2.31271	37
24	37190	2.68892	39190	2.55170	41217	2.42618	43274	2.31086	36
25	37223	2.68653	39223	2.54952	41251	2.42418	43308	2.30902	35
26	37256	2.68414	39257	2.54734	41285	2.42218	43343	2.30718	34
27	37289	2.68175	39290	2.54516	41319	2.42019	43378	2.30534	33
28	37322	2.67937	39324	2.54299	41353	2.41819	43412	2.30351	32
29	37355	2.67700	39357	2.54082	41387	2.41620	43447	2.30167	31
30	37388	2.67462	39391	2.53865	41421	2.41421	43481	2.29984	30
31	37422	2.67225	39425	2.53648	41455	2.41223	43516	2.29801	29
32	37455	2.66989	39458	2.53432	41490	2.41025	43550	2.29619	28
33	37488	2.66752	39492	2.53217	41524	2.40827	43585	2.29437	27
34	37521	2.66516	39526	2.53001	41558	2.40629	43620	2.29254	26
35	37554	2.66281	39559	2.52786	41592	2.40432	43654	2.29073	25
36	37588	2.66046	39593	2.52571	41626	2.40235	43689	2.28891	24
37	37621	2.65811	39626	2.52357	41660	2.40038	43724	2.28710	23
38	37654	2.65576	39660	2.52142	41694	2.39841	43758	2.28528	22
39	37687	2.65342	39694	2.51929	41728	2.39645	43793	2.28348	21
40	37720	2.65109	39727	2.51715	41763	2.39449	43828	2.28167	20
41	37754	2.64875	39761	2.51502	41797	2.39253	43862	2.27987	19
42	37787	2.64642	39795	2.51289	41831	2.39058	43897	2.27806	18
43	37820	2.64410	39829	2.51076	41865	2.38862	43932	2.27626	17
44	37853	2.64177	39862	2.50864	41899	2.38668	43966	2.27447	16
45	37887	2.63945	39896	2.50652	41933	2.38473	44001	2.27267	15
46	37920	2.63714	39930	2.50440	41968	2.38279	44036	2.27088	14
47	37953	2.63483	39963	2.50229	42002	2.38084	44071	2.26909	13
48	37986	2.63252	39997	2.50018	42036	2.37891	44105	2.26730	12
49	38020	2.63021	40031	2.49807	42070	2.37697	44140	2.26552	11
50	38053	2.62791	40065	2.49597	42105	2.37504	44175	2.26374	10
51	38086	2.62561	40098	2.49386	42139	2.37311	44210	2.26196	9
52	38120	2.62332	40132	2.49177	42173	2.37118	44244	2.26018	8
53	38153	2.62103	40166	2.48967	42207	2.36925	44279	2.25840	7
54	38186	2.61874	40200	2.48758	42242	2.36733	44314	2.25662	6
55	38220	2.61646	40234	2.48549	42276	2.36541	44349	2.25486	5
56	38253	2.61418	40267	2.48340	42310	2.36349	44384	2.25309	4
57	38286	2.61190	40301	2.48132	42345	2.36158	44418	2.25132	3
58	38320	2.60963	40335	2.47924	42379	2.35967	44453	2.24956	2
59	38353	2.60736	40369	2.47716	42413	2.35776	44488	2.24780	1
60	38386	2.60509	40403	2.47509	42447	2.35585	44523	2.24604	0
	69°		68°		67°		66°		
	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	

TABLE III. NATURAL TANGENTS AND COTANGENTS. 79									
°	24°		25°		26°		27°		°
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	44523	2.24604	46631	2.14451	48773	2.05030	50953	1.96261	60
1	44558	2.24428	46666	2.14288	48809	2.04879	50989	1.96120	59
2	44593	2.24252	46702	2.14125	48845	2.04728	51026	1.95979	58
3	44627	2.24077	46737	2.13963	48881	2.04577	51063	1.95838	57
4	44662	2.23902	46772	2.13801	48917	2.04426	51099	1.95698	56
5	44697	2.23727	46808	2.13639	48953	2.04276	51136	1.95557	55
6	44732	2.23553	46843	2.13477	48989	2.04125	51173	1.95417	54
7	44767	2.23378	46879	2.13316	49026	2.03975	51209	1.95277	53
8	44802	2.23204	46914	2.13154	49062	2.03825	51246	1.95137	52
9	44837	2.23030	46950	2.12993	49098	2.03675	51283	1.94997	51
10	44872	2.22857	46985	2.12832	49134	2.03526	51319	1.94858	50
11	44907	2.22683	47021	2.12671	49170	2.03376	51356	1.94718	49
12	44942	2.22510	47056	2.12511	49206	2.03227	51393	1.94579	48
13	44977	2.22337	47092	2.12350	49242	2.03078	51430	1.94440	47
14	45012	2.22164	47128	2.12190	49278	2.02929	51467	1.94301	46
15	45047	2.21992	47163	2.12030	49315	2.02780	51503	1.94162	45
16	45082	2.21819	47199	2.11871	49351	2.02631	51540	1.94023	44
17	45117	2.21647	47234	2.11711	49387	2.02483	51577	1.93885	43
18	45152	2.21475	47270	2.11552	49423	2.02335	51614	1.93746	42
19	45187	2.21304	47305	2.11392	49459	2.02187	51651	1.93608	41
20	45222	2.21132	47341	2.11233	49495	2.02039	51688	1.93470	40
21	45257	2.20961	47377	2.11075	49532	2.01891	51724	1.93332	39
22	45292	2.20790	47412	2.10916	49568	2.01743	51761	1.93195	38
23	45327	2.20619	47448	2.10758	49604	2.01596	51798	1.93057	37
24	45362	2.20449	47483	2.10600	49640	2.01449	51835	1.92920	36
25	45397	2.20278	47519	2.10442	49677	2.01302	51872	1.92782	35
26	45432	2.20108	47555	2.10284	49713	2.01155	51909	1.92645	34
27	45467	2.19938	47590	2.10126	49749	2.01008	51946	1.92508	33
28	45502	2.19769	47626	2.09969	49786	2.00862	51983	1.92371	32
29	45537	2.19599	47662	2.09811	49822	2.00715	52020	1.92235	31
30	45573	2.19430	47698	2.09654	49858	2.00569	52057	1.92098	30
31	45608	2.19261	47733	2.09498	49894	2.00423	52094	1.91962	29
32	45643	2.19092	47769	2.09341	49931	2.00277	52131	1.91826	28
33	45678	2.18923	47805	2.09184	49967	2.00131	52168	1.91690	27
34	45713	2.18755	47840	2.09028	50004	1.99986	52205	1.91554	26
35	45748	2.18587	47876	2.08872	50040	1.99841	52242	1.91418	25
36	45784	2.18419	47912	2.08716	50076	1.99695	52279	1.91282	24
37	45819	2.18251	47948	2.08560	50113	1.99550	52316	1.91147	23
38	45854	2.18084	47984	2.08405	50149	1.99406	52353	1.91012	22
39	45889	2.17916	48019	2.08250	50185	1.99261	52390	1.90876	21
40	45924	2.17749	48055	2.08094	50222	1.99116	52427	1.90741	20
41	45960	2.17582	48091	2.07939	50258	1.98972	52464	1.90607	19
42	45995	2.17416	48127	2.07785	50295	1.98828	52501	1.90472	18
43	46030	2.17249	48163	2.07630	50331	1.98684	52538	1.90337	17
44	46065	2.17083	48198	2.07476	50368	1.98540	52575	1.90203	16
45	46101	2.16917	48234	2.07321	50404	1.98396	52613	1.90069	15
46	46136	2.16751	48270	2.07167	50441	1.98253	52650	1.89935	14
47	46171	2.16585	48306	2.07014	50477	1.98110	52687	1.89801	13
48	46206	2.16420	48342	2.06860	50514	1.97966	52724	1.89667	12
49	46242	2.16255	48378	2.06706	50550	1.97823	52761	1.89533	11
50	46277	2.16090	48414	2.06553	50587	1.97680	52798	1.89400	10
51	46312	2.15925	48450	2.06400	50623	1.97538	52836	1.89266	9
52	46348	2.15760	48486	2.06247	50660	1.97395	52873	1.89133	8
53	46383	2.15596	48521	2.06094	50696	1.97253	52910	1.89000	7
54	46418	2.15432	48557	2.05942	50733	1.97111	52947	1.88867	6
55	46454	2.15268	48593	2.05790	50769	1.96969	52984	1.88734	5
56	46489	2.15104	48629	2.05637	50806	1.96827	53022	1.88602	4
57	46525	2.14940	48665	2.05485	50843	1.96685	53059	1.88469	3
58	46560	2.14777	48701	2.05333	50879	1.96544	53096	1.88337	2
59	46595	2.14614	48737	2.05182	50916	1.96402	53134	1.88205	1
60	46631	2.14451	48773	2.05030	50953	1.96261	53171	1.88073	0
	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	
	65°		64°		63°		62°		

	28°		29°		30°		31°		
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	53171	1.88073	55431	1.80405	57735	1.73205	60086	1.66428	60
1	53208	1.87941	55469	1.80281	57774	1.73089	60126	1.66318	59
2	53246	1.87809	55507	1.80158	57813	1.72973	60165	1.66209	58
3	53283	1.87677	55545	1.80034	57851	1.72857	60205	1.66099	57
4	53320	1.87546	55583	1.79911	57890	1.72741	60245	1.65990	56
5	53358	1.87415	55621	1.79788	57929	1.72625	60284	1.65881	55
6	53395	1.87283	55659	1.79665	57968	1.72509	60324	1.65772	54
7	53432	1.87152	55697	1.79542	58007	1.72393	60364	1.65663	53
8	53470	1.87021	55736	1.79419	58046	1.72278	60403	1.65554	52
9	53507	1.86891	55774	1.79296	58085	1.72163	60443	1.65445	51
10	53545	1.86760	55812	1.79174	58124	1.72047	60483	1.65337	50
11	53582	1.86630	55850	1.79051	58162	1.71932	60522	1.65228	49
12	53620	1.86499	55888	1.78929	58201	1.71817	60562	1.65120	48
13	53657	1.86369	55926	1.78807	58240	1.71702	60602	1.65011	47
14	53694	1.86239	55964	1.78685	58279	1.71588	60642	1.64903	46
15	53732	1.86109	56003	1.78563	58318	1.71473	60681	1.64795	45
16	53769	1.85979	56041	1.78441	58357	1.71358	60721	1.64687	44
17	53807	1.85850	56079	1.78319	58396	1.71244	60761	1.64579	43
18	53844	1.85720	56117	1.78198	58435	1.71129	60801	1.64471	42
19	53882	1.85591	56156	1.78077	58474	1.71015	60841	1.64363	41
20	53920	1.85462	56194	1.77955	58513	1.70901	60881	1.64256	40
21	53957	1.85333	56232	1.77834	58552	1.70787	60921	1.64148	39
22	53995	1.85204	56270	1.77713	58591	1.70673	60960	1.64041	38
23	54032	1.85075	56309	1.77592	58631	1.70560	61000	1.63934	37
24	54070	1.84946	56347	1.77471	58670	1.70446	61040	1.63826	36
25	54107	1.84818	56385	1.77351	58709	1.70332	61080	1.63719	35
26	54145	1.84689	56424	1.77230	58748	1.70219	61120	1.63612	34
27	54183	1.84561	56462	1.77110	58787	1.70106	61160	1.63505	33
28	54220	1.84433	56500	1.76990	58826	1.69992	61200	1.63398	32
29	54258	1.84305	56539	1.76869	58865	1.69879	61240	1.63292	31
30	54296	1.84177	56577	1.76749	58904	1.69766	61280	1.63185	30
31	54333	1.84049	56616	1.76630	58944	1.69653	61320	1.63079	29
32	54371	1.83922	56654	1.76510	58983	1.69541	61360	1.62972	28
33	54409	1.83794	56693	1.76390	59022	1.69428	61400	1.62866	27
34	54446	1.83667	56731	1.76271	59061	1.69316	61440	1.62760	26
35	54484	1.83540	56769	1.76151	59101	1.69203	61480	1.62654	25
36	54522	1.83413	56808	1.76032	59140	1.69091	61520	1.62548	24
37	54560	1.83286	56846	1.75913	59179	1.68979	61561	1.62442	23
38	54597	1.83159	56885	1.75794	59218	1.68866	61601	1.62336	22
39	54635	1.83033	56923	1.75675	59258	1.68754	61641	1.62230	21
40	54673	1.82906	56962	1.75556	59297	1.68643	61681	1.62125	20
41	54711	1.82780	57000	1.75437	59336	1.68531	61721	1.62019	19
42	54748	1.82654	57039	1.75319	59375	1.68419	61761	1.61914	18
43	54786	1.82528	57078	1.75200	59415	1.68308	61801	1.61808	17
44	54824	1.82402	57116	1.75082	59454	1.68196	61842	1.61703	16
45	54862	1.82276	57155	1.74964	59494	1.68085	61882	1.61598	15
46	54900	1.82150	57193	1.74846	59533	1.67974	61922	1.61493	14
47	54938	1.82025	57232	1.74728	59573	1.67863	61962	1.61388	13
48	54975	1.81899	57271	1.74610	59612	1.67752	62003	1.61283	12
49	55013	1.81774	57309	1.74492	59651	1.67641	62043	1.61179	11
50	55051	1.81649	57348	1.74375	59691	1.67530	62083	1.61074	10
51	55089	1.81524	57386	1.74257	59730	1.67419	62124	1.60970	9
52	55127	1.81399	57425	1.74140	59770	1.67309	62164	1.60865	8
53	55165	1.81274	57464	1.74022	59809	1.67198	62204	1.60761	7
54	55203	1.81150	57503	1.73905	59849	1.67088	62245	1.60657	6
55	55241	1.81025	57541	1.73788	59888	1.66978	62285	1.60553	5
56	55279	1.80901	57580	1.73671	59928	1.66867	62325	1.60449	4
57	55317	1.80777	57619	1.73555	59967	1.66757	62366	1.60345	3
58	55355	1.80653	57657	1.73438	60007	1.66647	62406	1.60241	2
59	55393	1.80529	57696	1.73321	60046	1.66538	62446	1.60137	1
60	55431	1.80405	57735	1.73205	60086	1.66428	62487	1.60033	0
	61°		60°		59°		58°		
	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	

TABLE III. NATURAL TANGENTS AND COTANGENTS.

81

°	32°		33°		34°		35°		°
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	62487	1.60033	64941	1.53986	67451	1.48256	70021	1.42815	60
1	62527	1.59930	64982	1.53888	67493	1.48163	70064	1.42726	59
2	62568	1.59826	65023	1.53791	67536	1.48070	70107	1.42638	58
3	62608	1.59723	65065	1.53693	67578	1.47977	70151	1.42550	57
4	62649	1.59620	65106	1.53595	67620	1.47885	70194	1.42462	56
5	62689	1.59517	65148	1.53497	67663	1.47792	70238	1.42374	55
6	62730	1.59414	65189	1.53400	67705	1.47699	70281	1.42286	54
7	62770	1.59311	65231	1.53302	67748	1.47607	70325	1.42198	53
8	62811	1.59208	65272	1.53205	67790	1.47514	70368	1.42110	52
9	62852	1.59105	65314	1.53107	67832	1.47422	70412	1.42022	51
10	62892	1.59002	65355	1.53010	67875	1.47330	70455	1.41934	50
11	62933	1.58900	65397	1.52913	67917	1.47238	70499	1.41847	49
12	62973	1.58797	65438	1.52816	67960	1.47146	70542	1.41759	48
13	63014	1.58695	65480	1.52719	68002	1.47053	70586	1.41672	47
14	63055	1.58593	65521	1.52622	68045	1.46962	70629	1.41584	46
15	63095	1.58490	65563	1.52525	68088	1.46870	70673	1.41497	45
16	63136	1.58388	65604	1.52429	68130	1.46778	70717	1.41409	44
17	63177	1.58286	65646	1.52332	68173	1.46686	70760	1.41322	43
18	63217	1.58184	65688	1.52235	68215	1.46595	70804	1.41235	42
19	63258	1.58083	65729	1.52139	68258	1.46503	70848	1.41148	41
20	63299	1.57981	65771	1.52043	68301	1.46411	70891	1.41061	40
21	63340	1.57879	65813	1.51946	68343	1.46320	70935	1.40974	39
22	63380	1.57778	65854	1.51850	68386	1.46229	70979	1.40887	38
23	63421	1.57676	65896	1.51754	68429	1.46137	71023	1.40800	37
24	63462	1.57575	65938	1.51658	68471	1.46046	71066	1.40714	36
25	63503	1.57474	65980	1.51562	68514	1.45955	71110	1.40627	35
26	63544	1.57372	66021	1.51466	68557	1.45864	71154	1.40540	34
27	63584	1.57271	66063	1.51370	68600	1.45773	71198	1.40454	33
28	63625	1.57170	66105	1.51275	68642	1.45682	71242	1.40367	32
29	63666	1.57069	66147	1.51179	68685	1.45592	71285	1.40281	31
30	63707	1.56969	66189	1.51084	68728	1.45501	71329	1.40195	30
31	63748	1.56868	66230	1.50988	68771	1.45410	71373	1.40109	29
32	63789	1.56767	66272	1.50893	68814	1.45320	71417	1.40022	28
33	63830	1.56667	66314	1.50797	68857	1.45229	71461	1.39936	27
34	63871	1.56566	66356	1.50702	68900	1.45139	71505	1.39850	26
35	63912	1.56466	66398	1.50607	68942	1.45049	71549	1.39764	25
36	63953	1.56366	66440	1.50512	68985	1.44958	71593	1.39679	24
37	63994	1.56265	66482	1.50417	69028	1.44868	71637	1.39593	23
38	64035	1.56165	66524	1.50322	69071	1.44778	71681	1.39507	22
39	64076	1.56065	66566	1.50228	69114	1.44688	71725	1.39421	21
40	64117	1.55966	66608	1.50133	69157	1.44598	71769	1.39336	20
41	64158	1.55866	66650	1.50038	69200	1.44508	71813	1.39250	19
42	64199	1.55766	66692	1.49944	69243	1.44418	71857	1.39165	18
43	64240	1.55666	66734	1.49849	69286	1.44329	71901	1.39079	17
44	64281	1.55567	66776	1.49755	69329	1.44239	71946	1.38994	16
45	64322	1.55467	66818	1.49661	69372	1.44149	71990	1.38909	15
46	64363	1.55368	66860	1.49566	69416	1.44060	72034	1.38824	14
47	64404	1.55269	66902	1.49472	69459	1.43970	72078	1.38738	13
48	64446	1.55170	66944	1.49378	69502	1.43881	72122	1.38653	12
49	64487	1.55071	66986	1.49284	69545	1.43792	72166	1.38568	11
50	64528	1.54972	67028	1.49190	69588	1.43703	72211	1.38484	10
51	64569	1.54873	67071	1.49097	69631	1.43614	72255	1.38399	9
52	64610	1.54774	67113	1.49003	69675	1.43525	72299	1.38314	8
53	64652	1.54675	67155	1.48909	69718	1.43436	72344	1.38229	7
54	64693	1.54576	67197	1.48816	69761	1.43347	72388	1.38145	6
55	64734	1.54478	67239	1.48722	69804	1.43258	72432	1.38060	5
56	64775	1.54379	67282	1.48629	69847	1.43169	72477	1.37976	4
57	64817	1.54281	67324	1.48536	69891	1.43080	72521	1.37891	3
58	64858	1.54183	67366	1.48442	69934	1.42992	72565	1.37807	2
59	64899	1.54085	67409	1.48349	69977	1.42903	72610	1.37722	1
60	64941	1.53986	67451	1.48256	70021	1.42815	72654	1.37638	0
°	32°		33°		34°		35°		°
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
57°									
56°									
55°									
54°									

	36°		37°		38°		39°		
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	72654	1.37638	75355	1.32704	78129	1.27994	80978	1.23490	60.
1	72699	1.37554	75401	1.32624	78175	1.27917	81027	1.23416	59
2	72743	1.37470	75447	1.32544	78222	1.27841	81075	1.23343	58
3	72788	1.37386	75492	1.32464	78266	1.27764	81123	1.23270	57
4	72832	1.37302	75538	1.32384	78316	1.27688	81171	1.23196	56
5	72877	1.37218	75584	1.32304	78363	1.27611	81220	1.23123	55
6	72921	1.37134	75629	1.32224	78410	1.27535	81268	1.23050	54
7	72966	1.37050	75675	1.32144	78457	1.27458	81316	1.22977	53
8	73010	1.36967	75721	1.32064	78504	1.27382	81364	1.22904	52
9	73055	1.36883	75767	1.31984	78551	1.27306	81413	1.22831	51
10	73100	1.36800	75812	1.31904	78598	1.27230	81461	1.22758	50
11	73144	1.36716	75858	1.31825	78645	1.27153	81510	1.22685	49
12	73189	1.36633	75904	1.31745	78692	1.27077	81558	1.22612	48
13	73234	1.36549	75950	1.31666	78739	1.27001	81606	1.22539	47
14	73278	1.36466	75996	1.31586	78786	1.26925	81655	1.22467	46
15	73323	1.36383	76042	1.31507	78834	1.26849	81703	1.22394	45
16	73368	1.36300	76088	1.31427	78881	1.26774	81752	1.22321	44
17	73413	1.36217	76134	1.31348	78928	1.26698	81800	1.22249	43
18	73457	1.36133	76180	1.31269	78975	1.26622	81849	1.22176	42
19	73502	1.36051	76226	1.31190	79022	1.26546	81898	1.22104	41
20	73547	1.35968	76272	1.31110	79070	1.26471	81946	1.22031	40
21	73592	1.35885	76318	1.31031	79117	1.26395	81995	1.21959	39
22	73637	1.35802	76364	1.30952	79164	1.26319	82044	1.21886	38
23	73681	1.35719	76410	1.30873	79212	1.26244	82092	1.21814	37
24	73726	1.35637	76456	1.30795	79259	1.26169	82141	1.21742	36
25	73771	1.35554	76502	1.30716	79306	1.26093	82190	1.21670	35
26	73816	1.35472	76548	1.30637	79354	1.26018	82238	1.21598	34
27	73861	1.35389	76594	1.30558	79401	1.25943	82287	1.21526	33
28	73906	1.35307	76640	1.30480	79449	1.25867	82336	1.21454	32
29	73951	1.35224	76686	1.30401	79496	1.25792	82385	1.21382	31
30	73996	1.35142	76733	1.30323	79544	1.25717	82434	1.21310	30
31	74041	1.35060	76779	1.30244	79591	1.25642	82483	1.21238	29
32	74086	1.34978	76825	1.30166	79639	1.25567	82531	1.21166	28
33	74131	1.34896	76871	1.30087	79686	1.25492	82580	1.21094	27
34	74176	1.34814	76918	1.30009	79734	1.25417	82629	1.21023	26
35	74221	1.34732	76964	1.29931	79781	1.25343	82678	1.20951	25
36	74267	1.34650	77010	1.29853	79829	1.25268	82727	1.20879	24
37	74312	1.34568	77057	1.29775	79877	1.25193	82776	1.20808	23
38	74357	1.34487	77103	1.29696	79924	1.25118	82825	1.20736	22
39	74402	1.34405	77149	1.29618	79972	1.25044	82874	1.20665	21
40	74447	1.34323	77196	1.29541	80020	1.24969	82923	1.20593	20
41	74492	1.34242	77242	1.29463	80067	1.24895	82972	1.20522	19
42	74538	1.34160	77289	1.29385	80115	1.24820	83022	1.20451	18
43	74583	1.34079	77335	1.29307	80163	1.24746	83071	1.20379	17
44	74628	1.33998	77382	1.29229	80211	1.24672	83120	1.20308	16
45	74674	1.33916	77428	1.29152	80258	1.24597	83169	1.20237	15
46	74719	1.33835	77475	1.29074	80306	1.24523	83218	1.20166	14
47	74764	1.33754	77521	1.28997	80354	1.24449	83268	1.20095	13
48	74810	1.33673	77568	1.28919	80402	1.24375	83317	1.20024	12
49	74855	1.33592	77615	1.28842	80450	1.24301	83366	1.19953	11
50	74900	1.33511	77661	1.28764	80498	1.24227	83415	1.19882	10
51	74946	1.33430	77708	1.28687	80546	1.24153	83465	1.19811	9
52	74991	1.33349	77754	1.28610	80594	1.24079	83514	1.19740	8
53	75037	1.33268	77801	1.28533	80642	1.24005	83564	1.19669	7
54	75082	1.33187	77848	1.28456	80690	1.23931	83613	1.19599	6
55	75128	1.33107	77895	1.28379	80738	1.23858	83662	1.19528	5
56	75173	1.33026	77941	1.28302	80786	1.23784	83712	1.19457	4
57	75219	1.32946	77988	1.28225	80834	1.23710	83761	1.19387	3
58	75264	1.32865	78035	1.28148	80882	1.23637	83811	1.19316	2
59	75310	1.32785	78082	1.28071	80930	1.23563	83860	1.19246	1
60	75355	1.32704	78129	1.27994	80978	1.23490	83910	1.19175	0
	53°		52°		51°		50°		
	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	

TABLE III.

NATURAL TANGENTS AND COTANGENTS.

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°	40°		41°		42°		43°		°
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	83910	1.19175	86929	1.15037	90040	1.11061	93252	1.07237	60
1	83960	1.19105	86980	1.14969	90093	1.10996	93306	1.07174	59
2	84009	1.19035	87031	1.14902	90146	1.10931	93360	1.07112	58
3	84059	1.18964	87082	1.14834	90199	1.10867	93415	1.07049	57
4	84108	1.18894	87133	1.14767	90251	1.10802	93469	1.06987	56
5	84158	1.18824	87184	1.14699	90304	1.10737	93524	1.06925	55
6	84208	1.18754	87236	1.14632	90357	1.10672	93578	1.06862	54
7	84258	1.18684	87287	1.14565	90410	1.10607	93633	1.06800	53
8	84307	1.18614	87338	1.14498	90463	1.10543	93688	1.06738	52
9	84357	1.18544	87389	1.14430	90516	1.10478	93742	1.06676	51
10	84407	1.18474	87441	1.14363	90569	1.10414	93797	1.06613	50
11	84457	1.18404	87492	1.14296	90621	1.10349	93852	1.06551	49
12	84507	1.18334	87543	1.14229	90674	1.10285	93906	1.06489	48
13	84556	1.18264	87595	1.14162	90727	1.10220	93961	1.06427	47
14	84606	1.18194	87646	1.14095	90781	1.10156	94016	1.06365	46
15	84656	1.18125	87698	1.14028	90834	1.10091	94071	1.06303	45
16	84706	1.18055	87749	1.13961	90887	1.10027	94125	1.06241	44
17	84756	1.17986	87801	1.13894	90940	1.09963	94180	1.06179	43
18	84806	1.17916	87852	1.13828	90993	1.09899	94235	1.06117	42
19	84856	1.17846	87904	1.13761	91046	1.09834	94290	1.06056	41
20	84906	1.17777	87955	1.13694	91099	1.09770	94345	1.05994	40
21	84956	1.17708	88007	1.13627	91153	1.09706	94400	1.05932	39
22	85006	1.17638	88059	1.13561	91206	1.09642	94455	1.05870	38
23	85057	1.17569	88110	1.13494	91259	1.09578	94510	1.05809	37
24	85107	1.17500	88162	1.13428	91313	1.09514	94565	1.05747	36
25	85157	1.17430	88214	1.13361	91366	1.09450	94620	1.05685	35
26	85207	1.17361	88265	1.13295	91419	1.09386	94676	1.05624	34
27	85257	1.17292	88317	1.13228	91473	1.09322	94731	1.05562	33
28	85307	1.17223	88369	1.13162	91526	1.09258	94786	1.05501	32
29	85358	1.17154	88421	1.13096	91580	1.09195	94841	1.05439	31
30	85408	1.17085	88473	1.13029	91633	1.09131	94896	1.05378	30
31	85458	1.17016	88524	1.12963	91687	1.09067	94952	1.05317	29
32	85509	1.16947	88576	1.12897	91740	1.09003	95007	1.05255	28
33	85559	1.16878	88628	1.12831	91794	1.08940	95062	1.05194	27
34	85609	1.16809	88680	1.12765	91847	1.08876	95118	1.05133	26
35	85660	1.16741	88732	1.12699	91901	1.08813	95173	1.05072	25
36	85710	1.16672	88784	1.12633	91955	1.08749	95229	1.05010	24
37	85761	1.16603	88836	1.12567	92008	1.08686	95284	1.04949	23
38	85811	1.16535	88888	1.12501	92062	1.08622	95340	1.04888	22
39	85862	1.16466	88940	1.12435	92116	1.08559	95395	1.04827	21
40	85912	1.16398	88992	1.12369	92170	1.08496	95451	1.04766	20
41	85963	1.16329	89043	1.12303	92223	1.08432	95506	1.04705	19
42	86014	1.16261	89097	1.12238	92277	1.08369	95562	1.04644	18
43	86064	1.16192	89149	1.12172	92331	1.08306	95618	1.04583	17
44	86115	1.16124	89201	1.12106	92385	1.08243	95673	1.04522	16
45	86166	1.16056	89253	1.12041	92439	1.08179	95729	1.04461	15
46	86216	1.15987	89306	1.11975	92493	1.08116	95785	1.04401	14
47	86267	1.15919	89358	1.11909	92547	1.08053	95841	1.04340	13
48	86318	1.15851	89410	1.11844	92601	1.07990	95897	1.04279	12
49	86368	1.15783	89463	1.11778	92655	1.07927	95952	1.04218	11
50	86419	1.15715	89515	1.11713	92709	1.07864	96008	1.04158	10
51	86470	1.15647	89567	1.11648	92763	1.07801	96064	1.04097	9
52	86521	1.15579	89620	1.11582	92817	1.07738	96120	1.04036	8
53	86572	1.15511	89672	1.11517	92872	1.07676	96176	1.03976	7
54	86623	1.15443	89725	1.11452	92926	1.07613	96232	1.03915	6
55	86674	1.15375	89777	1.11387	92980	1.07550	96288	1.03855	5
56	86725	1.15308	89830	1.11321	93034	1.07487	96344	1.03794	4
57	86776	1.15240	89883	1.11256	93088	1.07425	96400	1.03734	3
58	86827	1.15172	89935	1.11191	93143	1.07362	96457	1.03674	2
59	86878	1.15104	89988	1.11126	93197	1.07299	96513	1.03613	1
60	86929	1.15037	90040	1.11061	93252	1.07237	96569	1.03553	0
°	40°		48°		47°		46°		°
	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	

